ALGEBRAS OF COMPACT OPERATORS

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CONTENTS

INTRODUCTION

The purpose of this thesis is to examine certain classes of bounded linear operators on a Banach space X in an algebraic light, i.e. as elements of a Banach algebra rather than as operators on X, the Banach algebra in general being the algebra B(X) of all bounded linear operators on X. We choose those properties which can be expressed in general algebraic terms, and then study elements of a general Banach algebra which satisfy these properties. The class originally chosen, suggested to me by Professor F. F. Bonsall, was the class of compact operators on X. As the algebraic properties of such operators generally involve their spectral properties, it was natural to extend our study to include Riesz operators as well.

In the case of Riesz operators the choice of a suitable algebraic characteristic is fairly obvious. We can express the definition of a Riesz operator as follows. If $T \in B(X)$ then T is a Riesz operator if and only if, for each element λ in $\mathrm{Sp}\, T \setminus \{0\}$, there exists a projection P_λ of finite rank which commutes with T and is such that $\lambda \notin \mathrm{Sp}(T - TP_\lambda)$ and $(T - \lambda I)^\nu P_\lambda = 0$ for some positive integer ν. We shall show later that we may remove the condition that $(T - \lambda I)^\nu P_\lambda = 0$. Thus we define an element a of a general Banach algebra A to be a Riesz element if, for each λ in $\mathrm{Sp}\, a \setminus \{0\}$, there exists an idempotent e_λ in the socle of A which commutes with a and is such that $\lambda \notin \mathrm{Sp}(a - ae_\lambda)$. We observe that the existence of a non-zero Riesz element does not necessarily imply that A has a socle since, if a has zero spectral radius, then a is a Riesz element.

A very different method for dealing with compact operators is provided by the following result, due to Vala (15). He proves that, if T_1 and T_2 are non-zero elements of $B(X)$, then the operator on $B(X)$ given by $S \longrightarrow T_1 S T_2 \quad (S \in B(X))$ is compact if and only if both T_1 and T_2 are compact operators. Denoting this operator by ${}_1\mathcal{T}_2$, we extend this result in Chapter I to cover the case of Riesz operators. We prove that ${}_1\mathcal{T}_2$ is a quasi-nilpotent operator if and only if either T_1 or T_2 is quasi-nilpotent, and that ${}_1\mathcal{T}_2$ is a Riesz operator with non-zero spectral radius if and only if both T_1 and T_2 are Riesz operators with non-zero spectral radii. We may complete this form of characterisation by stating that, if T_1 and T_2 are non-zero elements of $B(X)$, then ${}_1\mathcal{T}_2$ has finite rank if and only if both T_1 and T_2 have finite rank. In particular, if ${}_1\mathcal{T}_1$ is the operator on $B(X)$ given by $S \rightarrow T_1 S T_1 \quad (S \in B(X))$, then ${}_1\mathcal{T}_1$ is a finite rank, compact, quasi-nilpotent, Riesz operator resp. if and only if T_1 is a finite rank, compact, quasi-nilpotent, Riesz operator. We complete Chapter I by expressing the spectral properties of ${}_1\mathcal{T}_2$ in terms of those of T_1 and T_2.

We now set these ideas in the context of a general Banach algebra. For elements a and c in A we define the operator ${}_aT_c$ on A by

$$ {}_aT_c b = abc \qquad (b \in A). $$

In Chapter III we study an element a for which ${}_aT_a$ is either a compact or a Riesz operator. Our main result is a generalisation of the characterisation of Riesz operators given in Chapter I; if A is semi-simple, then a is a Riesz element of A if and only if ${}_aT_a$ is a Riesz operator. The existence of the socle of A plays a large part in this work, and we may characterise the socle of a semi-simple Banach algebra A in terms of ${}_aT_a$ as follows. If $a \neq 0$, then ${}_aT_a$ has finite rank if and only if the socle of A exists and contains a. We obtain partial results when ${}_aT_c$ is a Riesz operator for distinct elements a and c of A, and we show that ${}_aT_c$ is a Riesz operator when a and c are

Riesz elements of a semi-simple Banach algebra A. In this case we can obtain a complete description of the spectral properties of ${}_aT_c$ in terms of those of a and c.

The assumption that ${}_aT_a$ is compact does not appear to yield significantly more information about the element a than is obtained under the assumption that ${}_aT_a$ is only a Riesz operator. However the set of all compact operators on a Banach space X does form a Banach algebra, and hence, instead of considering individual elements a of a Banach algebra A such that ${}_aT_a$ is compact, we consider a Banach algebra A for which ${}_aT_a$ is compact for each element a of A. We call such an algebra a compact Banach algebra, and study them in Chapter V. Our main results are a representation theorem which shows that a primitive compact Banach algebra may be represented as an algebra of compact operators, and a structure theorem, showing that the structure space of a compact Banach algebra is discrete under the usual hull-kernel topology. These two theorems allow us to represent completely a compact B* algebra; a primitive compact B* algebra is isometrically isomorphic to the algebra of all compact operators on some Hilbert space, and a compact B* algebra is isometrically isomorphic to a $B(\infty)$ sum of such algebras.

It is perhaps worth observing that, if the well-known conjecture holds that a compact operator on a Banach space X is the limit in the operator norm of a sequence of finite rank operators, then we have an immediate algebraic characterisation for compact operators; T is a compact operator on X if and only if T lies in the closure of the socle of B(X).

The remaining two chapters, Chapter II and Chapter IV, are both short. In Chapter II we study a Banach algebra A for which left multiplication by some element a is a Riesz operator. This is mainly preparatory work for the next chapter, but we are able to considerably generalise a result by Kaplansky on completely continuous algebras. Such an algebra is a particular case of a compact algebra, and the results obtained in Chapter V for a compact Banach algebra may be compared with those obtained by Kaplansky for a completely continuous algebra. Chapter IV contains various representation theorems for a primitive Banach algebra which contains elements a and c for which ${}_aT_c$ is either a compact operator or a Riesz operator.

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References and Notation.

The reader is assumed to have a knowledge of the basic elements of functional analysis, particularily of the theory of compact operators and Banach algebras. The reader is referred to Zaanen (18) for the theory of compact operators, and to Rickart (12) for Banach algebra theory. In Chapter I we give a brief account of the theory of Riesz operators; the details may be found in two recent papers by West, (16) and (17). Most of the results of this thesis are unpublished, but the bulk of Chapter V, together with several results from earlier chapters, can be found in a paper by the author (1), which is to appear in the Proceedings of the London Mathematical Society.

One small result concerning Banach algebras which we shall often use, but is not found explicitly in Rickart (12), is the following.

This is more or less a special case of the theorem: $L_1 L_2 \subset P$, P primitive ideal $\Rightarrow$ $L_1 \subset P$ or $L_2 \subset P$, — see (12) Theorem 2.2.9 (iv).

If a and c are elements of a primitive Banach algebra A such that $aAc = (0)$, then either $a = 0$ or $c = 0$. The proof consists of taking a faithful strictly irreducible representation $a \longrightarrow S_a$ of A on a Banach space X, and observing that if $a \neq 0$, $c \neq 0$, there exist x and y in X such that $S_a x \neq 0$ and $S_c y \neq 0$. Since the representation is strictly irreducible there exists an element b in A such that $S_b S_c y = x$, and thus $S_a S_b S_c y \neq 0$. Thus $abc \neq 0$.

The notation that we shall use is consistent with that in (12). We do, however, make special mention of the following points.

We shall always consider Banach algebras over the complex field, which we denote by $\mathbb{C}$.

Given an element a of a Banach algebra A, we denote by A(a) the least closed subalgebra of A containing a. Then A(a) will be the closure of the set of all polynomials in a without constant term.

Given an element a of a Banach algebra A, we write $Sp_A a$ and $\nu_A(a)$ for the spectrum of a and the spectral radius of a respectively. Where there is no confusion concerning which algebra we are considering, we write Sp a and $\nu(a)$.

For a Banach space X we denote by B(X) the algebra of all bounded linear operators on X, having the norm $\|-\|$, and by X* the space of all bounded linear functionals on X. If $x \in X$, $y \in X^*$, we denote by (x,y) the value of y at x, and by $x \otimes y$ the bounded linear operator on X given by $x \otimes y(u) = (u,y)\, x$, $(u \in X)$. If $T \in B(X)$ we denote by T* the adjoint of T on X*, where $(Tx,y) = (x,T^*y)$, $(x \in X,\ y \in X^*)$.

If $T \in B(X)$ and M is a closed subspace of X such that $T(M) \subseteq M$, we denote by T/M the operator T restricted to M.

If $T \in B(X)$ we denote by Null T and Range T the subspaces $\{x;\ x \in X \text{ and } Tx = 0\}$ and $\{Tx;\ x \in X\}$ respectively.

The reference system which we use is as follows: (Z.Y.X) refers to object X in section Y of Chapter Z, the object usually being a theorem, lemma, corollary, or definition. Within Chapter Z we just refer to (Y.X). References to the bibliography are given in round brackets.

I. COMPACT OPERATORS AND RIESZ OPERATORS.

1. Introduction.

In this chapter we consider two classes of operators on a Banach space —— compact operators and Riesz operators. We characterise these two classes in terms of similar classes of operators on the algebra $B(X)$ of all bounded linear operators on the Banach space X. If T is a bounded linear operator on X, we prove that the operator

$$S \to TST \qquad (S \in B(X))$$

on $B(X)$ is compact (Riesz) if and only if T is compact (Riesz). In fact we prove rather more, considering operators of the type

$$S \to T_1 S T_2 \qquad (S \in B(X)).$$

This is done in § 3.

Since Riesz, and hence compact, operators have very strong spectral properties, it is natural to ask whether any relationship exists between the spectral properties of T_1 and T_2 and those of the derived operator $S \to T_1 S T_2 \quad (S \in B(X))$. In §4 we investigate this problem and show that such a relationship exists and that it is a very strong one.

We begin in §2 with the definitions and some properties of compact and Riesz operators that we shall need. We include three proofs at the end of the section, since we later produce analogues of these results for general Banach algebras. For the proofs of the others we refer the reader to Zaanen (18) for compact operators, and to West (16), (17) for Riesz operators. In addition the reader might consult a recent paper by Bonsall (3) in which the spectral properties of a compact operator are obtained by Banach algebra techniques.

2. Definitions and Standard Properties.

We suppose X is a Banach space and denote by X_1 the unit ball in X.

DEFINITION 2.1. An operator $T \in B(X)$ is said to be a compact operator if and only if $T(X_1)$ is relatively compact in X under the norm topology, i.e. the closure of $T(X_1)$ is a compact set.

An equivalent definition is: $T \in B(X)$ is compact if and only if, for each bounded sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ converges.

THEOREM 2.2. The set of compact operators in $B(X)$ forms a closed two-sided ideal of $B(X)$, which contains all finite rank operators.

We denote by $K(X)$ the ideal of all compact operators on X.

THEOREM 2.3. $T \in K(X)$ if and only if $T^* \in K(X^*)$.

If $T \in B(X)$, denote by Null T and Range T the sets $\{x;\ x \in X \text{ and } Tx = 0\}$ and $\{Tx;\ x \in X\}$ respectively.

THEOREM 2.4. (Riesz-Schauder)

Suppose that $T^p \in K(X)$ for some positive integer p and that λ is a non-zero point in Sp T. Then there exists an integer $\nu(\lambda)$ which is the smallest integer such that Null $(T-\lambda I)^{\nu(\lambda)}$ = Null $(T-\lambda I)^{\nu(\lambda)+1}$ and is also the smallest integer such that Range$(T-\lambda I)^{\nu(\lambda)}$ = Range $(T-\lambda I)^{\nu(\lambda)+1}$. In addition, Null $(T-\lambda I)^{\nu(\lambda)}$ has finite dimension, Range $(T-\lambda I)^{\nu(\lambda)}$ is closed, and X = Null $(T-\lambda I)^{\nu(\lambda)} \oplus$ Range $(T-\lambda I)^{\nu(\lambda)}$.

Since $T^{*p} \in K(X^*)$ and $\lambda \in$ Sp T^*, a similar integer $\nu^*(\lambda)$ exists for T^* and $\nu(\lambda) = \nu^*(\lambda)$.

$\nu(\lambda)$ is called the index of λ for T. For ease of notation we shall in future just write ν for the index, it being clear from the context with which λ and T we are concerned.

Theorem 2.4 leads us to the definition of a Riesz operator.

DEFINITION 2.5. An operator T in B(X) is said to be a Riesz operator if and only if, for each non-zero element λ of Sp T, there exist closed subspaces N_λ and R_λ of X such that

(i) N_λ and R_λ are invariant under T

(ii) $X = N_\lambda \oplus R_\lambda$

(iii) N_λ has finite dimension

(iv) $T - \lambda I$ restricted to N_λ is nilpotent

(v) $T - \lambda I$ restricted to R_λ is a homeomorphism.

In fact we can identify N_λ and R_λ as follows. If we take ν to be the smallest integer such that $(T - \lambda I|N_\lambda)^\nu = 0$, then it is easily seen that ν is the smallest integer such that Null $(T - \lambda I)^\nu$ = Null $(T - \lambda I)^{\nu+1}$ and is also the smallest integer such that Range$(T - \lambda I)^\nu$ = Range$(T - \lambda I)^{\nu+1}$. Then N_λ = Null $(T - \lambda I)^\nu$ and R_λ = Range$(T - \lambda I)^\nu$.

We denote by R(X) the set of Riesz operators on X. We observe that R(X) contains all quasi-nilpotent operators on X and, by Theorem 2.4, all operators T in B(X) for which T^p is compact for some positive integer p.

We now describe the Ruston characterisation of R(X), see (13) and (16). The quotient algebra B(X)/K(X) is a Banach algebra under the usual infimum norm. Let $T \to [T]$ be the canonical mapping of B(X) onto B(X)/K(X).

THEOREM 2.6. $T \in R(X)$ if and only if $[T]$ is a quasi-nilpotent element in B(X)/K(X).

Since $\nu([T]) = \lim \|[T]^n\|^{1/n} = \inf \|[T]^n\|^{1/n}$ we have;

COROLLARY 2.7. The following statements are equivalent;

(i) $T \in R(X)$

(ii) given $\epsilon > 0$, there exists an integer N and operators $C_n \in K(X)$ $(n \geqslant N)$ such that

$$\|T^n - C_n\| \leqslant \varepsilon^n$$

(iii) given $\varepsilon > 0$, there exists an integer N and an operator $C_N \in K(X)$ such that

$$\|T^N - C_N\| \leqslant \varepsilon^N.$$

Theorem 2.6 yields the following results, which may be compared with Theorem 2.2.

THEOREM 2.8. (i) If $T_1, T_2 \in R(X)$ and $T_1T_2 = T_2T_1$, then $T_1 + T_2 \in R(X)$.

(ii) If $T_1 \in B(X)$, $T_2 \in R(X)$ and $T_1T_2 = T_2T_1$, then $T_1T_2 \in R(X)$.

(iii) If $T_n \in R(X)$ $(n = 1,2,\ldots)$ and if $T_n \to T$ so that $TT_n = T_nT$ $(n = 1,2,\ldots)$, then $T \in R(X)$.

(iv) If $T \in R(X)$, then $T^* \in R(X^*)$.

NOTE. In general $R(X)$ is not closed under addition, multiplication, or under the uniform topology —— see the counter examples in § 5, (16).

THEOREM 2.9. The identity operator I lies in $R(X)$ if and only if X has finite dimension.

THEOREM 2.10. If $T \in R(X)$, Sp T is countable with 0 as the only possible accumulation point.

THEOREM 2.11. If $T \in R(X)$ and has a closed invariant subspace M, then T/M is a Riesz operator on M.

If T is a Riesz operator and $\lambda \in \mathrm{Sp}\, T \setminus \{0\}$, denote by P_λ the projection onto N_λ. We observe that $TP_\lambda = P_\lambda T$. Since we have a homeomorphism between N_λ and X/R_λ it is clear that P_λ lies in $B(X)$.

THEOREM 2.12. (i) $N_\lambda \cap N_\mu = (0)$ if $\lambda \neq \mu$.

(ii) $P_\lambda P_\mu = 0$ if $\lambda \neq \mu$.

(iii) $\mathrm{Null}\, T \cap N_\lambda = (0)$.

Proof. (i) If $\lambda \neq \mu$, the polynomials $(t - \lambda)^{\nu(\lambda)}$ and $(t - \mu)^{\nu(\mu)}$ in the complex variable t are relatively prime. Hence there exist polynomials $p(t)$ and $q(t)$ such that

$$p(t)\,(t - \lambda)^{\nu(\lambda)} + q(t)\,(t - \mu)^{\nu(\mu)} = 1.$$

Thus $p(T)\,(T - \lambda I)^{\nu(\lambda)} + q(T)\,(T - \mu I)^{\nu(\mu)} = I$ and hence it follows that, if $x \in N_\lambda \cap N_\mu$, then

$$x = p(T).\,(T - \lambda I)^{\nu(\lambda)}x + q(T).\,(T - \mu I)^{\nu(\mu)}x = 0.$$

(ii) For any $x \in X$, $P_\lambda P_\mu x$ lies in N_λ. We also have

$$(T - \mu I)^{\nu(\mu)} P_\lambda P_\mu x = P_\lambda (T - \mu I)^{\nu(\mu)} P_\mu x = 0$$

as $TP_\lambda = P_\lambda T$. Hence if $\lambda \neq \mu$, $P_\lambda P_\mu x \in N_\lambda \cap N_\mu = (0)$ and so $P_\lambda P_\mu = 0$.

(iii) If $x \in \mathrm{Null}\, T \cap N_\lambda$, then $Tx = 0$ and $(T - \lambda I)^{\nu(\lambda)} x = 0$. Hence $x = 0$, by expanding $(T - \lambda I)^{\nu(\lambda)}$!

THEOREM 2.13. Let $T \in R(X)$ and suppose $\lambda_1, \ldots, \lambda_n$ are distinct non-zero points in $\mathrm{Sp}\, T$. Denote by T' and T'' the operators $T(I - \sum P_{\lambda_i})$ and $T(\sum P_{\lambda_i})$ respectively. Then

(i) $\mathrm{Sp}\, T' \setminus \{0\} = \mathrm{Sp}\, T \setminus \{\lambda_1, \ldots, \lambda_n, 0\}$

(ii) $\mathrm{Sp}\, T'' \setminus \{0\} = \{\lambda_1, \ldots, \lambda_n\}$.

Proof. (i) By Theorem 2.8 (ii) both T' and T'' lie in $R(X)$. Let $P = \sum P_{\lambda_i}$ and $Q = I - P$. Then $P^2 = P$ and $Q^2 = Q$ by Theorem 2.12.

If $\lambda \in \mathrm{Sp}\, T' \setminus \{0\}$, λ is an eigen value of T' since T' lies in $R(X)$. Thus there exists $x \in X$, $x \neq 0$, such that $TQx = \lambda x$. Hence $Qx \neq 0$ and

$\lambda Qx = QTQx = TQ^2x = TQx$. Thus $\lambda \in \mathrm{Sp}\, T \setminus \{0\}$. Since $P_{\lambda_i} Q = QP_{\lambda_i} = 0$ $(1 \leq i \leq n)$ it follows that $P_{\lambda_i} Qx = 0$, and hence $\lambda \neq \lambda_i$ $(1 \leq i \leq n)$ since $Qx \in \mathrm{Null}\,(T - \lambda I)^{\nu}$. Thus $\lambda \in \mathrm{Sp}\, T \setminus \{\lambda_1, \ldots, \lambda_n, 0\}$.

If $\lambda \in \mathrm{Sp}\, T \setminus \{\lambda_1, \ldots, \lambda_n, 0\}$, there exists $x \in X$, $x \neq 0$, such that $Tx = \lambda x$. $P_{\lambda_i} x = 0$ $(1 \leq i \leq n)$ by Theorem 2.12, and hence it follows that $Qx = x$ and $T'x = \lambda x$. Thus $\lambda \in \mathrm{Sp}\, T' \setminus \{0\}$.

(ii) For each i $(1 \leq i \leq n)$ there exists $x_i \in X$, $x_i \neq 0$, such that $Tx_i = \lambda_i x_i$. By Theorem 2.12 $Px_i = x_i$ and so $T''x_i = \lambda_i x_i$. Hence $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathrm{Sp}\, T''$.

If $\lambda \in \mathrm{Sp}\, T'' \setminus \{0\}$, there exists $x \in X$, $x \neq 0$, such that $T''x = \lambda x$, since $T'' \in R(X)$. Then $T(\sum P_{\lambda_i} x) = \lambda x$ and so $P_{\lambda_i} x \neq 0$ for some i. Then $\lambda P_{\lambda_i} x = P_{\lambda_i} T''x = TP_{\lambda_i} Px = TP_{\lambda_i} x$ by Theorem 2.12. Hence $\lambda \in \mathrm{Sp} T \setminus \{0\}$ and so $P_{\lambda}(P_{\lambda_i} x) = P_{\lambda_i} x$. Thus by Theorem 2.12. it follows that $\lambda = \lambda_i$, and hence $\mathrm{Sp}\, T'' \setminus \{0\} = \{\lambda_1, \ldots, \lambda_n\}$.

We may now express the concept of a Riesz operator in a more algebraic form

THEOREM 2.14. If T lies in $B(X)$, then T is a Riesz operator if and only if, for each $\lambda \in \mathrm{Sp}\, T \setminus \{0\}$, there exists an idempotent P_λ in the socle of $B(X)$ that commutes with T and is such that

(i) there exists an integer ν such that $(T - \lambda I)^{\nu} P_\lambda = 0$

(ii) $\lambda \notin \mathrm{Sp}(T - TP_\lambda)$.

Proof. If T is a Riesz operator the result follows from Theorem 2.13. Conversely, let $N_\lambda = P_\lambda(X)$ and $R_\lambda = (I - P_\lambda)(X)$. We only have to verify that $T - \lambda I | R_\lambda$ is a homeomorphism. If S is the inverse of $\lambda^{-1}(T - TP_\lambda) - I$, then $P_\lambda S = -P_\lambda = SP_\lambda$ and so $S(R_\lambda) \subseteq R_\lambda$. Hence $\lambda^{-1} S | R_\lambda$ is the inverse of $T - \lambda I | R_\lambda$ and our proof is complete.

NOTE. In fact we can remove condition (i), as will be seen in Corollary 3.4.3.

3. The operator ${}_1\mathcal{T}_2$.

We suppose T_1 and T_2 are bounded linear operators on a Banach space X.

DEFINITION 3.1. We denote by ${}_1\mathcal{T}$, $\mathcal{T}_1$, ${}_1\mathcal{T}_2$ respectively the bounded linear operators on B(X) given by

$$
\begin{aligned}
{}_1\mathcal{T}S &= T_1S \qquad (S \in B(X)) \\
\mathcal{T}_1S &= ST_1 \qquad (S \in B(X)) \\
{}_1\mathcal{T}_2S &= T_1ST_2 \qquad (S \in B(X)).
\end{aligned}
$$

LEMMA 3.2. (i) $\|{}_1\mathcal{T}\| = \|T_1\|, \|\mathcal{T}_1\| = \|T_1\|, \|{}_1\mathcal{T}_2\| = \|T_1\|\,\|T_2\|$.
(ii) $\nu({}_1\mathcal{T}) = \nu(T_1), \nu(\mathcal{T}_1) = \nu(T_1), \nu({}_1\mathcal{T}_2) = \nu(T_1)\,\nu(T_2)$.

Proof. (i) Clearly $\|{}_1\mathcal{T}_2\| \leqslant \|T_1\|\,\|T_2\|$. To prove equality we proceed as follows. Given $\varepsilon > 0$, there exists $x \in X$, $y \in X^*$, such that $\|T_1x\| \geqslant \|T_1\|(1-\varepsilon)$, $\|T_2^*y\| \geqslant \|T_2^*\|(1-\varepsilon) = \|T_2\|(1-\varepsilon)$, $\|x\| = 1$, and $\|y\| = 1$. Then we have $\|x \otimes y\| = \|x\|\,\|y\| = 1$ and $\|T_1(x \otimes y)T_2\| = \|T_1x\|\,\|T_2^*y\| \geqslant (1-\varepsilon)^2\|T_1\|\,\|T_2\|$. Hence $\|{}_1\mathcal{T}_2\| = \|T_1\|\,\|T_2\|$.

Similarily we may prove $\|{}_1\mathcal{T}\| = \|T_1\| = \|\mathcal{T}_1\|$.

(ii) Since ${}_1\mathcal{T}_2^{\,n}S = T_1^n\,ST_2^n$, it follows from (i) that

$$\|{}_1\mathcal{T}_2^{\,n}\| = \|T_1^n\|\,\|T_2^n\|. \text{ Hence } \nu({}_1\mathcal{T}_2) = \nu(T_1).\nu(T_2).$$

Similarily we have $\nu({}_1\mathcal{T}) = \nu(T_1) = \nu(\mathcal{T}_1)$.

The following theorem is crucial for the work in this chapter and indeed initiated the work contained in this thesis. It is a corollary of a theorem by Vala — see (15) Theorem 3. The proof that we shall give is due to Bonsall and is slightly easier, although essentially the same. We need the following form of Ascoli's theorem.

Let E be a compact space and Y a complete metric space. Let $\{f_n\}$ be an equicontinuous sequence of mappings of E into Y, such that, for each x in E, the set $\{f_n(x): n = 1,2,\ldots\}$ is contained in a compact subset of Y. Then there exists a subsequence $\{f_{n_k}\}$ which converges uniformly in E.

THEOREM 3.3. If T_1, T_2 are non-zero members of $B(X)$, then ${}_1\mathcal{T}_2$ is a compact operator on $B(X)$ if and only if both T_1 and T_2 are compact.

Proof. We suppose T_1 and T_2 are compact. Let X_1 be the closed unit ball in X and put $E = \overline{T_2(X_1)}$, so that E is compact in the norm topology. Given a sequence $\{S_n\}$ in $B(X)$ with $\|S_n\| \leq 1$, consider the sequence $\{T_1S_n\}$ mapping E into X. It is equicontinuous since $\|T_1S_nx - T_1S_nx'\| \leq \|T_1\|\|x - x'\|$, and if $x \in E$ then $\{T_1S_nx;\ n = 1,2,\ldots\}$ lies in $\overline{T_1X_2}$ which is compact, where X_2 is the closed ball of radius $\|T_2\|$ in X. Thus the conditions of Ascoli's theorem are satisfied, and so there exists a subsequence $\{T_1S_{n_k}\}$ converging uniformly on E. Hence given $\varepsilon > 0$, there exists K such that

$$\|T_1S_{n_k}x - T_1S_{n_{k'}}x\| \leq \varepsilon \quad (k,\ k' \geq K,\ x \in E).$$

Hence we have

$$\|T_1S_{n_k}T_2x - T_1S_{n_{k'}}T_2x\| \leq \varepsilon \qquad (k,\ k' \geq K,\ x \in X_1)$$

giving

$$\|T_1S_{n_k}T_2 - T_1S_{n_{k'}}T_2\| \leq \varepsilon \qquad (k,\ k' \geq K).$$

Thus $\{T_1S_{n_k}T_2\}$ converges and hence ${}_1\mathcal{T}_2$ is compact.

Conversely, suppose ${}_1\mathcal{T}_2$ is compact. Since $T_2 \neq 0$ we may find y_2 in X^* such that $T_2^*y_2 \neq 0$. By considering ${}_1\mathcal{T}_2$ restricted to operators of the form $x \otimes y_2$ it is clear that T_1 must be compact. Similarily we may prove T_2^* is compact, and hence, by Theorem 2.3, T_2 is compact.

NOTE. The operators ${}_1\mathcal{T}$ and $\mathcal{T}_1$ are not compact unless either $T_1 = 0$ or X is finite-dimensional. For if $T_1 \neq 0$ take x_1 in X such that $T_1x_1 \neq 0$, and consider ${}_1\mathcal{T}$ restricted to the subspace of operators of the form $x_1 \otimes y$ $(y \in X^*)$. Then ${}_1\mathcal{T}$ compact implies that the identity operator on X^* is compact and hence X^*, and so also X, is finite -dimensional. We may similarily treat the case when $\mathcal{T}_1$ is compact.

If $T_1 = 0$, clearly ${}_1\mathcal{T} = \mathcal{T}_1 = 0$. If X is finite-dimensional then so also is B(X) and hence ${}_1\mathcal{T}$ and $\mathcal{T}_1$ are compact.

We now prove an analagous theorem to Theorem 3.3 for Riesz operators.

THEOREM 3.4. If T_1 and T_2 are Riesz operators, then ${}_1\mathcal{T}_2$ is a Riesz operator.

Proof. From Corollary 2.7 it follows that, given $\varepsilon > 0$, there exists an integer n and compact operators C_1 and C_2 in B(X) such that

$$\|T_1^n - C_1\| < \varepsilon^n, \quad \|T_2^n - C_2\| < \varepsilon^n.$$

Denote by ${}_1\mathcal{C}_2$ the operator on B(X) given by

$${}_1\mathcal{C}_2(S) = C_1SC_2 \qquad (S \in B(X)).$$

By Theorem 3.3 ${}_1\mathcal{C}_2$ is compact. Since ${}_1\mathcal{T}_2^n(S) = T_1^nST_2^n$ we have

$$\begin{aligned}\|{}_1\mathcal{T}_2^n(S) - {}_1\mathcal{C}_2(S)\| &= \|(T_1^n - C_1)ST_2^n + C_1S(T_2^n - C_2)\| \\ &\leq \varepsilon^n(\|T_1^n\| + \|T_2^n\| + \varepsilon^n)\,\|S\| \\ &\leq 3M^n\varepsilon^n\|S\|\end{aligned}$$

where $M = \max\,(\|T_1\|, \|T_2\|, \varepsilon)$. Hence $\|{}_1\mathcal{T}_2^n - {}_1\mathcal{C}_2\| \leq 3\,M^n\varepsilon^n$ and hence by Corollary 2.7 ${}_1\mathcal{T}_2$ is a Riesz operator on B(X).

We now prove a converse theorem.

THEOREM 3.5. If $T_1, T_2 \in B(X)$ then

(i) either T_1 or T_2 is quasi-nilpotent if ${}_1\mathcal{T}_2$ is quasi-nilpotent

(ii) both T_1 and T_2 must be Riesz operators if ${}_1\mathcal{T}_2$ is a Riesz operator but is not a quasi-nilpotent operator.

Proof. (i) This follows immediately from Lemma 3.2.

(ii) Suppose ${}_1\mathcal{T}_2$ is a Riesz operator and λ is a non-zero eigen value of ${}_1\mathcal{T}_2$. Take $S \in \text{Null}({}_1\mathcal{T}_2 - \lambda I)$, $S \neq 0$. Then $T_1^n S \in \text{Null}({}_1\mathcal{T}_2 - \lambda I)$ for all n, and, since $\text{Null}({}_1\mathcal{T}_2 - \lambda I)$ is finite-dimensional, it follows that there exists an integer N such that $\{S, T_1 S, \ldots, T_1^N S\}$ are linearly dependent. Thus there exists a non-zero polynomial $p(T_1)$ in T_1 such that $p(T_1)S = 0$. We may write this in the form

$$\prod_{i=1}^{m}(T_1 - \mu_i I)T_1^n S = 0 \qquad (\mu_i \neq 0,\ 1 \leq i \leq m).$$

Since $T_1^n S T_2^n = \lambda^n S$ it follows that $T_1^n S \neq 0$, and so for some i $\text{Null}(T_1 - \mu_i I) \neq (0)$. Hence there exists some complex number $\lambda_1 \neq 0$ and a point $x_1 \in X$, $x_1 \neq 0$, such that $T_1 x_1 = \lambda_1 x_1$.

Similarily there exists a complex number $\lambda_2 \neq 0$ and a point $y_2 \in X^*$, $y_2 \neq 0$, such that $T_2^* y_2 = \lambda_2 y_2$.

Denote by U the closed subspace of $B(X)$ consisting of all operators of the form $x \boxtimes y_2$. This is invariant under ${}_1\mathcal{T}_2$ since

$${}_1\mathcal{T}_2(x \boxtimes y_2) = T_1 x \boxtimes T_2^* y_2 = \lambda_2 T_1 x \boxtimes y_2.$$

The map $x \longrightarrow x \boxtimes y_2$ $(x \in X)$ is a homeomorphism between X and U, and hence there is an algebraic homeomorphism between $B(X)$ and $B(U)$ in which T_1 corresponds to $1/\lambda_2 \cdot {}_1\mathcal{T}_2|U$. Since ${}_1\mathcal{T}_2|U$ is a Riesz operator on U by Theorem 2.11, it follows that T_1 is a Riesz operator.

In a similar way, by considering the subspace V of $B(X)$ consisting of all operators of the form $x_1 \boxtimes y$, we can identify V and X^*, and so prove that T_2^* is a Riesz operator. Then by Theorems 2.8 (iv) and 2.11 it follows that T_2 is a Riesz operator.

COROLLARY 3.6. T_1 is a compact, quasi-nilpotent, Riesz operator respectively if and only if ${}_1\mathcal{T}_1$ is a compact, quasi-nilpotent, Riesz operator respectively.

NOTE (i) The operator ${}_1\mathcal{T}\ (\mathcal{T}_1)$ is a Riesz operator only if either T_1 is quasi-nilpotent or X has finite dimension. In the later case ${}_1\mathcal{T}$ and $\mathcal{T}_1$ will be compact. For by Lemma 3.2 $\nu({}_1\mathcal{T}) = \nu(\mathcal{T}_1) = \nu(T_1)$. If ${}_1\mathcal{T}$ is not quasi-nilpotent, we can prove as in Theorem 3.5 that T_1 has a non-zero eigen value, λ_1 say, such that $T_1 x_1 = \lambda_1 x_1$ for some $x_1 \in X$, $x_1 \neq 0$. Then all operators of the form $x_1 \otimes y$ lie in the null space of ${}_1\mathcal{T} - \lambda I$ which can only have finite dimension if X* is finite-dimensional, i.e. if X is finite dimensional. Similarly we may deal with $\mathcal{T}_1$.

(ii) If $T_1 \in R(X)$ and B is a closed subalgebra of B(X) whose centre contains T_1, the operator

$$S \longrightarrow T_1 S \qquad (S \in B)$$

is a Riesz operator on B. This follows from the fact that the square of this operator is ${}_1\mathcal{T}_1 / B$ and hence is a Riesz operator.

This result may be compared with the equivalent result for compact operators by Bonsall, (3) Theorem 1.

4. The Spectral Properties of ${}_1\mathcal{T}_2$.

We assume throughout this section that $T_1, T_2 \in R(X)$.

THEOREM 4.1. $\lambda \in \mathrm{Sp}\ {}_1\mathcal{T}_2 \setminus \{0\}$ if and only if there exists $\mu \in \mathrm{Sp}\ T_2 \setminus \{0\}$ such that $\lambda/\mu \in \mathrm{Sp}\ T_1$.

Proof. Take $\lambda \in \mathrm{Sp}\ {}_1\mathcal{T}_2 \setminus \{0\}$. As in the proof of Theorem 3.5 (ii), if $S \in \mathrm{Null}\ ({}_1\mathcal{T}_2 - \lambda I)$, there exists a polynomial $q(T_2)$ such that $S\, q(T_2) = 0$. $ST_2^n \neq 0$ if $S \neq 0$ since $T_1^n S T_2^n = \lambda^n S$, and so we may write

$$ST_2^n \prod_{i=1}^{m} (T_2 - \mu_i I) = 0$$

for some non-zero μ_i. Hence there exists a non-zero operator S' in

Null $({}_1\mathcal{T}_2 - \lambda I)$ and $\mu \in \mathrm{Sp}\, T_2 \setminus \{0\}$ such that

$$S'T_2 = \mu S'.$$

Then we have

$$T_1 S' = 1/\mu\, T_1 S' T_2 = \lambda/\mu\, S'$$

and thus $\lambda/\mu \in \mathrm{Sp}\, T_1$.

Conversely, suppose $\lambda/\mu \in \mathrm{Sp}\, T_1 \setminus \{0\}$ and $\mu \in \mathrm{Sp}\, T_2 \setminus \{0\}$. Since T_1 and T_2^* are Riesz operators, there exists $x_1 \in X$, $x_1 \neq 0$, and $y_2 \in X^*$, $y_2 \neq 0$, such that $T_1 x_1 = \lambda/\mu\, x_1$ and $T_2^* y_2 = \mu y_2$. Then $x_1 \otimes y_2 \neq 0$ and ${}_1\mathcal{T}_2(x_1 \otimes y_2) = \lambda(x_1 \otimes y_2)$. Thus $\lambda \in \mathrm{Sp}\, {}_1\mathcal{T}_2$.

NOTATION. If $\mu \in \mathrm{Sp}\, T_1 \setminus \{0\}$, denote by P_μ the projection onto usually associated with T_1 Null $(T_1 - \mu I)^\nu$, and if $\mu \in \mathrm{Sp}\, T_2 \setminus \{0\}$, denote by Q_μ the projection onto Null $(T_2 - \mu I)^\nu$ associated with T_2.

Since T_1 and T_2 are Riesz operators, ${}_1\mathcal{T}_2$ is a Riesz operator. If $\lambda \in \mathrm{Sp}\, {}_1\mathcal{T}_2 \setminus \{0\}$, let $B(X) = N_\lambda \oplus R_\lambda$ be the corresponding decomposition of $B(X)$, and let $\mathcal{P}_\lambda$ be the projection onto N_λ.

THEOREM 4.2. Let $\{\mu_1, \dots, \mu_n\} = \{\mu ;\ \mu \in \mathrm{Sp}\, T_2 \setminus \{0\} \text{ and } \lambda/\mu \in \mathrm{Sp}\, T_1 \setminus \{0\}\}$ Then

$$\mathcal{P}_\lambda(S) = \sum_1^n P_{\lambda/\mu_i} S\, Q_{\mu_i} \qquad (S \in B(X)).$$

Proof. We first prove that, if $S \in N_\lambda$, then $S = \sum_1^n P_{\lambda/\mu_i} S\, Q_{\mu_i}$.

Let $T_3 = T_2(I - Q_{\mu_1} - \dots - Q_{\mu_n})$. Then $T_3 \in R(X)$ by Theorem 2.8. If $S \in N_\lambda$ and $SQ_{\mu_i} = 0$ $(1 \leq i \leq n)$ then

$$({}_1\mathcal{T}_3 - \lambda I)^\nu S = \big(({}_1\mathcal{T}_2 - \lambda I)^\nu S\big)\big(I - Q_{\mu_1} - \dots - Q_{\mu_n}\big) = 0.$$

By Theorem 2.13, $\mathrm{Sp}\, T_3 \setminus \{0\} = \mathrm{Sp}\, T_2 \setminus \{\mu_1, \dots \mu_n, 0\}$, and thus λ is not an eigen value of ${}_1\mathcal{T}_3$ by Theorem 4.1. Hence

$$S \in N_\lambda,\ SQ_{\mu_i} = 0,\ (1 \leq i \leq n) \Longrightarrow S = 0. \qquad \text{------------ (1)}$$

Now take any S in N_λ. Then $S(I - Q_{\mu_1} - \dots - Q_{\mu_n}) \in N_\lambda$ since

$$\begin{aligned}({}_1\mathcal{T}_2 - \lambda I)^\nu \big(S(I - Q_{\mu_1} - \dots - Q_{\mu_n})\big) &= \big(({}_1\mathcal{T}_2 - \lambda I)^\nu S\big)\big(I - Q_{\mu_1} - \dots - Q_{\mu_n}\big)\\ &= 0.\end{aligned}$$

But $S(I - Q_{\mu_1} - \dots - Q_{\mu_n})Q_{\mu_i} = 0$ by Theorem 2.12, and so by (1)

$$S \in N_\lambda \implies S = \sum_1^n SQ_{\mu_i} \qquad \text{---------------} \quad (2)$$

Similarly we have

$$S \in N_\lambda \implies S = \sum_1^n P_{\lambda/\mu_i} S \qquad \text{---------------} \quad (3)$$

Now take some fixed i in $1 \leq i \leq n$ and define T_4 by $T_4 = T_2 Q_{\mu_i}$. Then $T_4 \in R(X)$ by Theorem 2.8. If $S \in N_\lambda$ then

$$({}_1T_4 - \lambda I)^\nu (SQ_{\mu_i}) = \left(({}_1T_2 - \lambda I)^\nu S\right) Q_{\mu_i} = 0.$$

By Theorem 2.13 $\mathrm{Sp}\ T_4 \setminus \{0\} = \{\mu_i\}$ and thus, by applying (3) to the operator ${}_1T_4$ we have $SQ_{\mu_i} = P_{\lambda/\mu_i} SQ_{\mu_i}$. Hence

$$S \in N_\lambda \implies S = \sum_{i=1}^n P_{\lambda/\mu_i} SQ_{\mu_i} \qquad \text{-------------} \quad (4)$$

To obtain the converse result we proceed as follows. We choose some fixed element $\mu \in \{\mu_1, \dots, \mu_n\}$ and prove that any operator of the form $P_{\lambda/\mu} SQ_\mu$ lies in N_λ. Operators of this form have finite rank and form a finite-dimensional subspace V of B(X), since $P_{\lambda/\mu} SQ_\mu$ has range in $P_{\lambda/\mu}(X)$, while its adjoint has range in $Q^*_\mu(X^*)$. ${}_1T_2$ maps V into itself since

$$T_1 P_{\lambda/\mu} SQ_\mu T_2 = P_{\lambda/\mu} T_1 S T_2 Q_\mu,$$

and λ is clearly an eigen value of ${}_1T_2|V$. Now λ is the only eigen value, since suppose $\lambda' \neq \lambda$, $\lambda' \neq 0$, and $T_1 S' T_2 = \lambda' S'$ for some $S' \in V$, $S' \neq 0$. Then λ' is an eigen value of ${}_1T_2$ and so by (4)

$$S' = \sum_1^m P_{\lambda'/\rho_i} S' Q_{\rho_i}$$

for some elements $\rho_1, \dots, \rho_m$ in $\mathrm{Sp}\ T_2 \setminus \{0\}$, such that $\lambda'/\rho_1, \dots, \lambda'/\rho_m$ lie in $\mathrm{Sp}\ T_1 \setminus \{0\}$. As $S' \in V$, we also have $S' = P_{\lambda/\mu} S' Q_\mu$. Since the equalities $\lambda/\mu = \lambda'/\rho_i$, $\mu = \rho_i$, can not both hold for each i, it follows from Theorem 2.12 that

$$S' = \sum_{i=1}^m P_{\lambda/\mu} P_{\lambda'/\rho_i} S' Q_{\rho_i} Q_\mu = 0.$$

Similarily 0 is not an eigen value of ${}_1T_2|V$, since, if $T_1 S' T_2 = 0$, $S' \in V$, then $S' T_2 = 0$ as $\mathrm{Null}\ T_1 \cap \mathrm{Null}(T_1 - \lambda/\mu I)^\nu = (0)$ by Theorem 2.12. Since $S' Q_\mu = S'$ and $(T_2 - \mu I)^\nu Q_\mu = 0$, it follows that

$$0 = S'(T_2 - \mu I)^{\nu} Q_{\mu} = (-\mu)^{\nu} S' Q_{\mu} = (-\mu)^{\nu} S'.$$

Hence ${}_1\mathcal{T}_2|V$ is a bounded linear operator on a finite-dimensional Banach space with λ as its only eigen value. Such an operator can be represented by an upper diagonal matrix whose entries along the main diagonal are all λ — see (6) page 107. Then for some integer ν $({}_1\mathcal{T}_2 - \lambda I)^{\nu} V = (0)$, and so

$$P_{\lambda/\mu} S Q_{\mu} \in N_{\lambda} \qquad \text{for all} \quad S \in B(X).$$

Hence

$$S \in N_{\lambda} \Longleftrightarrow S = \sum_{i}^{n} P_{\lambda/\mu_i} S Q_{\mu_i}$$

If $S \in R_{\lambda}$, suppose $S = ({}_1\mathcal{T}_2 - \lambda I)^{\nu} S'$. Then

$$\sum_{i}^{n} P_{\lambda/\mu_i} S Q_{\mu_i} = ({}_1\mathcal{T}_2 - \lambda I)^{\nu} \left(\sum_{i}^{n} P_{\lambda/\mu_i} S' Q_{\mu_i} \right) = 0$$

since $\sum P_{\lambda/\mu_i} S' Q_{\mu_i} \in N_{\lambda}$. Thus

$$\mathcal{P}_{\lambda}(S) = \sum_{i}^{n} P_{\lambda/\mu_i} S Q_{\mu_i}.$$

NOTE. We observe that $\mathcal{P}_{\lambda}$ is an operator of the type we have been considering, being the sum of a finite family of operators which are obtained by left and right multiplication.

We remark that the verification that the set

$$\{\mu;\ \mu \in \mathrm{Sp}\ T_2 \setminus \{0\} \ \text{ and } \ \lambda/\mu \in \mathrm{Sp}\ T_1 \setminus \{0\}\}$$

is finite is easy. For, if μ is in this set, then $|\mu| \geq |\lambda|/\nu(T_1)$, and hence, since $\mathrm{Sp}\ T_2$ is countable with 0 as the only possible accumulation point, it follows that the set is finite.

II. THE OPERATORS ${}_aT$ AND T_a.

1. Introduction.

In this chapter we consider a Banach algebra **A** and the operators ${}_aT$ and T_a on it, given by left and right multiplication by an element a on A.

In § 2 we are concerned to show how, when ${}_aT$ is a Riesz operator on A, the spectrum of ${}_aT$ relates to that of a. We show that, if $\lambda \in \mathbb{C}$, $\lambda \neq 0$, then $\lambda \in \mathrm{Sp}\ {}_aT \setminus \{0\}$ if and only if $\lambda \in \mathrm{Sp}\ a \setminus \{0\}$, and that if so, there exists a non-zero idempotent e_λ commuting with a such that ${}_{e_\lambda}T$ is the projection onto $\mathrm{Null}({}_aT - \lambda I)^{\bar{\nu}}$. In this section we often use the convenient notational device of apparently introducing a unit element in such expressions as $(a - \lambda 1)^{\nu} b$.

In § 3 we show that, if A is a primitive Banach algebra and ${}_aT$ is a Riesz operator for some element a of **A** with $\nu(a) > 0$, then A is homeomorphic to a full finite-dimensional matrix algebra. This generalises a result of Kaplansky for completely continuous algebras and helps to answer a problem raised by Olubummo in connection with such algebras.

2. ${}_aT$ and T_a.

Let **A** be a Banach algebra. We do not assume that **A** has a unit element or is commutative.

DEFINITION 2.1. For $a \in A$ denote by ${}_aT$ and T_a the operators on A given by

$${}_aT\, b = ab \qquad (b \in A)$$
$$T_a\, b = ba \qquad (b \in A).$$

We shall be discussing elements a of A for which ${}_aT$ is a compact or Riesz operator. We first examine the relationship between Sp a and Sp ${}_aT$. This has been done, when ${}_aT$ is compact, by Bonsall, (3) Lemma 1 and Theorem 2. He works with a commutative algebra with a unit element, but his proofs do not use these assumptions. We shall obtain equivalent results for the case when ${}_aT$ is a Riesz operator, but this requires rather more preparation.

LEMMA 2.2. If $a \in A$, then $\nu({}_aT) = \nu(T_a) = \nu(a)$.

Proof. Since $\|a^n b\| \leq \|a^n\| \|b\|$ $(b \in A)$ it is clear that $\|{}_aT^n\| \leq \|a^n\|$ and so $\nu({}_aT) \leq \nu(a)$. However, $\|{}_aT^n\| \geq \|a^{n+1}\|/\|a\|$. Since $\|a^{n+1}\|^{1/n}/\|a\|^{1/n} \to \nu(a)$ as $n \to \infty$ it follows that $\nu({}_aT) \geq \nu(a)$ and hence $\nu({}_aT) = \nu(a)$. Similarily we have $\nu(T_a) = \nu(a)$.

The following lemma is suggested by (3) Theorem 3.

LEMMA 2.3. Let a be an element of A such that ${}_aT$ is a Riesz operator on A. If $\lambda \in \mathrm{Sp}\,{}_aT \setminus \{0\}$, let $A = N_\lambda \oplus R_\lambda$ be the corresponding decomposition of A. Then there exists a non-zero idempotent e_λ, commuting with a, such that ${}_{e_\lambda}T$ is the projection onto N_λ associated with ${}_aT$.

Proof. For some integer ν, $N_\lambda = \mathrm{Null}({}_aT - \lambda I)^\nu$ and $R_\lambda = \mathrm{Range}({}_aT - \lambda I)^\nu$. Then N_λ and R_λ are closed right ideals of A.

Although A may not have a unit element, $(a - \lambda 1)^\nu - (-\lambda)^\nu 1$ can be interpreted in the obvious way as an element of A. Hence there exist elements $p \in N_\lambda$, $q \in R_\lambda$ such that

$$(a - \lambda 1)^\nu - (-\lambda)^\nu 1 = p + q.$$

If $b \in N_\lambda$, then $(a - \lambda 1)^\nu b = 0$, and hence $-(-\lambda)^\nu b = pb + qb$. Now $qb \in R_\lambda$ since $q \in R_\lambda$, and $pb \in N_\lambda$ since $p \in N_\lambda$. Thus $qb \in N_\lambda \cap R_\lambda = (0)$, giving $-(-\lambda)^\nu b = pb$. Writing $e_\lambda = -1/(-\lambda)^\nu.p$ we have $e_\lambda b = b$ for all b in N_λ. Conversely, if $b = e_\lambda b$, then $b \in N_\lambda$ since $e_\lambda \in N_\lambda$. Clearly $e_\lambda^2 = e_\lambda$, and $e_\lambda \neq 0$ since $N_\lambda \neq (0)$.

We now prove that $ae_\lambda = e_\lambda a$. Since $aN_\lambda \subset N_\lambda$, and $aR_\lambda \subset R_\lambda$, we have $ap \in N_\lambda$ and $aq \in R_\lambda$. Since we also have $pa \in N_\lambda$, $qa \in R_\lambda$, and $a(p + q) = (p + q)a$, it follows that $ap = pa$ and $aq = qa$. Hence $ae_\lambda = e_\lambda a$ as required.

Finally, if $b \in R_\lambda$, suppose $b = ({}_aT - \lambda I)^\nu c$. Then, since a and e_λ commute, ${}_{e_\lambda}T\, b = {}_{e_\lambda}T({}_aT - \lambda I)^\nu c = ({}_aT - \lambda I)^\nu (e_\lambda c) = 0$ as $e_\lambda \in N_\lambda$. Hence ${}_{e_\lambda}T$ is the projection onto N_λ.

THEOREM 2.4. Let a be an element of A such that ${}_aT$ is a Riesz operator on A. Then $Sp(a) \setminus \{0\} = Sp\,{}_aT \setminus \{0\}$.

Proof. Clearly $Sp(a) \setminus \{0\} \supseteq Sp\,{}_aT \setminus \{0\}$. Suppose $\lambda \in Sp(a) \setminus \{0\}$ but $\lambda \notin Sp\,{}_aT \setminus \{0\}$. Let $\{\lambda_1, \ldots, \lambda_n\} = \{\mu ;\ \mu \in Sp\,{}_aT,\ |\mu| \geq |\lambda|\}$. This is a finite set since the spectrum of a Riesz operator is countable, with 0 as its only possible accumulation point. Let $e_1, \ldots, e_n$ be the set of idempotents in A corresponding to $\lambda_1, \ldots, \lambda_n$ as in Lemma 2.3. Put $c = a - \sum_{i=1}^{n} ae_i$. It follows from Theorem 1.2.8 (ii) that ${}_cT$ is a Riesz operator, and from Theorem 1.2.13 (i) that $\nu({}_cT) < |\lambda|$. Thus, by Lemma 2.2, $\nu(c) < |\lambda|$.

By Theorem 1.2.12 ${}_{e_i}T\ {}_{e_j}T = 0$ $(i \neq j)$, and so

$e_i e_j = e_i e_j^2 = {}_{e_i}T\ {}_{e_j}T(e_j) = 0$. Hence there exists a commutative subalgebra B of A containing a, e_i, $(1 \leq i \leq n)$ such that $Sp_B a \setminus \{0\} = Sp_A a \setminus \{0\}$. Since $\nu_B(c) = \nu_A(c)$, it follows that $\lambda \notin Sp_B c$. Let $\hat{b}$ be the Gelfand transform of b in B, and let Φ be the carrier space of B. Since $(a - \lambda_i 1)^\nu e_i = 0$ $(1 \leq i \leq n)$ it follows that, if $\hat{e}_i(\phi) = 1$, then $\hat{a}(\phi) = \lambda_i$. Thus if $\hat{a}(\phi) = \lambda$, then $\hat{e}_i(\phi) = 0$ $(1 \leq i \leq n)$, since $\hat{e}_i(\phi)$ can not be 1, and hence $\hat{c}(\phi) = \lambda$. Since $\nu(c) < |\lambda|$, this is a contradiction, and so $Sp(a) \setminus \{0\} = Sp_a T \setminus \{0\}$.

COROLLARY 2.5. Let a be an element of **A** such that ${}_aT$ is a Riesz operator on A, and let $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then $\lambda \in \mathrm{Sp}\,{}_aT$ if and only if $\lambda \in \mathrm{Sp}\, a$, and if so, there exists a non-zero idempotent e_λ commuting with a such that ${}_{e_\lambda}T$ is the associated projection onto $\mathrm{Null}({}_aT - \lambda I)^{\nu}$.

3. Primitive Algebras and Completely Continuous Algebras.

In (7) Kaplansky makes the following definition.

DEFINITION 3.1. A Banach algebra **A** is said to be completely continuous (c.c) if, for each element $a \in A$, the operators on A given by

$$b \longrightarrow ab \qquad (b \in \mathbf{A})$$

$$b \longrightarrow ba \qquad (b \in \mathbf{A})$$

are both compact.

We may also define left (right) complete continuity in the obvious way.

In (7) Lemma 4 Kaplansky proves the following result.

THEOREM 3.2. **A** primitive Banach algebra is completely continuous if and only if it is homeomorphic to a full finite-dimensional matrix algebra.

We can generalise this result as follows.

THEOREM 3.3. Let **A** be a primitive Banach algebra with an element a such that $\nu(a) > 0$ and ${}_aT$ is a Riesz operator. Then **A** is homeomorphic to a full finite-dimensional matrix algebra, and hence is a completely continuous algebra.

Proof. Since $\nu(a) > 0$, it follows from Corollary 2.5 that there exists a non-zero idempotent e in A, commuting with a, such that eA is finite-dimensional. Consider the anti-homomorphism $a' \rightarrow R_{a'}$ from A into $B(eA)$ given by

$$R_{a'}(eb) = eba' \qquad (eb \in eA).$$

If $eAa' = 0$ then $a' = 0$, since $e \neq 0$ and A is primitive, and thus it is an anti-isomorphism. Hence A is finite-dimensional. Since A is primitive, it has a faithful, strictly irreducible, representation on some Banach space, which will be finite-dimensional. By (12) Corollary 2.4.7 the representation will be strictly dense, and hence A maps onto the whole matrix algebra. A matrix algebra has a unique topology, and hence A is homeomorphic to a full finite-dimensional matrix algebra. It follows from Theorem 3.2 that A is completely continuous.

This theorem answers, in the primitive case, a question raised by Olubummo in (11) where he asks whether a left c.c. algebra is a c.c algebra. The following example shows that this is untrue in general.

Take X to be an infinite-dimensional Banach space, and let f be some fixed non-zero element in X^*. Let A be the algebra of all operators on X of the form $x \otimes f$. If $\|x_n \otimes f\| \leq 1$, then

$$(x \otimes f)(x_n \otimes f) = f(x_n)\, x \otimes f$$

and $|f(x_n)| \leq 1$. It follows that left multiplication by $x \otimes f$ is a compact operator on A, and so A is a left c.c. algebra. If $x_1 \in X$ and $f(x_1) = 1$, then right multiplication by $x_1 \otimes f$ is the identity operator on A, and thus can not be compact, as X has infinite dimension.

In this example A has a "large" radical, and it still leaves open the question when A is semi-simple.

The other main theorem on c.c. Banach algebras is given by

Kaplansky in (8) Theorem 5.1 in which he proves that the structure space of a completely continuous Banach algebra is discrete, in the usual hull-kernel topology. We shall obtain this result as a corollary in a later chapter.

III THE OPERATOR ${}_aT_c$

1. Introduction.

If a and c are elements of a Banach algebra A, we define ${}_aT_c$ to be the bounded linear operator on A given by ${}_aT_c\, b = abc \quad (b \in A)$.

In § 2 we examine an element a of a Banach algebra A for which the operator ${}_aT_a$ is a Riesz operator. Using results obtained in Chapter II for ${}_aT$ we can show that a has the usual spectral properties associated with a Riesz operator, i.e. Sp a is countable with 0 as the only possible accumulation point; for each $\lambda \in \mathrm{Sp}\, a \setminus \{0\}$ there exists a non-zero idempotent e_λ such that $(a - \lambda 1)^\nu e_\lambda = 0$ for some integer ν, $ae_\lambda = e_\lambda a$, and $\lambda \notin \mathrm{Sp}(a - ae_\lambda)$. We show that these properties uniquely define e_λ, that e_λ lies in the smallest closed subalgebra containing a, and that in the case when $A = B(X)$ the idempotent e_λ corresponds to the usual null-space projection P_λ.

When A is semi-simple, we show how the idempotents e_λ give rise to minimal idempotents, and thus in § 3 we examine the socle of A. Our principal result is that, if a and c are non-zero elements in a primitive Banach algebra A, then ${}_aT_c$ has finite rank if and only if the socle of A exists and contains a and c. If a = c we can replace the condition that A be primitive by the condition that A be semi-simple.

In § 4 we produce a necessary and sufficent condition for ${}_aT_a$ to be a Riesz operator. We define an element a in A to be a Riesz element if, for each λ in $\mathrm{Sp}\, a \setminus \{0\}$, there exists an idempotent f_λ in the socle of A that commutes with a and is such that $\lambda \notin \mathrm{Sp}(a - af_\lambda)$. Then, if A is semi-simple and a and c are Riesz elements, ${}_aT_c$ is a Riesz operator. In particular, ${}_aT_a$ is a Riesz operator if and only if a is a Riesz element. When applied to the algebra B(X) we see that

T is a Riesz operator on X if and only if T is a Riesz element in $B(X)$. This allows us to remove some of the conditions in the definition of a Riesz operator. Perhaps the neatest expression of the definition is as follows; T is a Riesz operator if and only if, for each λ in $\mathrm{Sp}\, T \setminus \{0\}$, there exists a projection P_λ of finite rank, commuting with T, such that $\lambda \notin \mathrm{Sp}(T - TP_\lambda)$.

In § 5 we investigate the spectral properties of ${}_aT_c$. If a and c are Riesz elements then ${}_aT_c$ is a Riesz operator and we can identify $\mathrm{Sp}\, {}_aT_c$ and the projections onto $\mathrm{Null}({}_aT_c - \lambda I)^\nu$ for each λ in $\mathrm{Sp}\, {}_aT_c \setminus \{0\}$. These results are very similar to those obtained for Riesz operators in Chapter I, § 4.

2. The operator ${}_aT_a$.

DEFINITION 2.1. For elements a and c in a Banach algebra A denote by ${}_aT_c$ the bounded linear operator on A given by

$$ {}_aT_c\, b = abc \qquad (b \in A). $$

Note that if A is primitive, then ${}_aT_c = 0$ implies that either $a = 0$ or $c = 0$. If A is semi-simple, then ${}_aT_a = 0$ implies that $a = 0$.

We observe that $\nu({}_aT_c) \leq \nu(a)\,\nu(c)$ and that, since $\|{}_aT_a{}^n\| \geq \|a^{2n+1}\| / \|a\|$, $\nu({}_aT_a) = (\nu(a))^2 = \nu(a^2)$.

LEMMA 2.2. Let ${}_aT_a$ be a Riesz operator on A. Then

(i) ${}_aT$ restricted to a closed subalgebra of A whose centre contains a is Riesz,

(ii) if B is a closed subalgebra of A whose centre contains a then $\mathrm{Sp}_B a \setminus \{0\} = \mathrm{Sp}_A a \setminus \{0\} = \mathrm{Sp}({}_aT/B) \setminus \{0\}$,

(iii) Sp a is countable, with 0 as the only possible accumulation point,

(iv) for each $\lambda \in \mathrm{Sp}\, a \setminus \{0\}$ there exists a non-zero idempotent e_λ commuting with a such that, for some integer ν_λ, $(a - \lambda 1)^{\nu_\lambda} e_\lambda = 0$ and $e_\lambda e_\mu = 0$ if $\lambda \neq \mu$.

Proof. If B is a closed subalgebra of **A** whose centre contains a, then ${}_aT_a$ maps B into itself and, by Theorem 1.2.11, ${}_aT_a|B$ is a Riesz operator. But ${}_aT_a|B = {}_aT^2|B$ and thus by the Ruston characterisation ${}_aT|B$ is a Riesz operator.

(ii) Since $B \subseteq A$, $\mathrm{Sp}_B a \setminus \{0\} \supseteq \mathrm{Sp}_A a \setminus \{0\}$. If $\lambda \in \mathrm{Sp}_B a \setminus \{0\}$, then $\lambda \in \mathrm{Sp}({}_aT|B) \setminus \{0\}$ by Theorem 2.2.4, and hence there exists b in B, $b \neq 0$, such that $ab = \lambda b$, Hence $\lambda \in \mathrm{Sp}_A a \setminus \{0\}$ and the required result follows.

(iii) This follows immediately from (ii).

(iv) Taking B as in (ii) we obtain our required idempotents from Corollary 2.2.5. By Theorem 1.2.12 ${}_{e_\lambda}T\ {}_{e_\mu}T = 0$ $(\lambda \neq \mu)$ and hence $e_\lambda e_\mu = e_\lambda e_\mu^{\,2} = {}_{e_\lambda}T\ {}_{e_\mu}T(e_\mu) = 0$.

COROLLARY 2.3. If T is a Riesz operator on a Banach space X, then left multiplication by T on the centraliser of T is a Riesz operator.

Proof. By definition, the centraliser of T is the set of all bounded linear operators on X which commute with T. This is a closed subalgebra of B(X) and hence, by Theorem 1.3.4, we have the required result.

We may compare this result with that by Bonsall, (3) Theorem 1. He proves that, if T is a compact operator on X, then left multiplication by T on its centraliser is a compact operator.

We remind the reader that left multiplication by T is not a Riesz operator on $B(X)$ unless either T is quasi-nilpotent or X is finite-dimensional.

The following theorem may be compared with Theorem 1.2.13.

THEOREM 2.4. If ${}_aT_a$ is a Riesz operator on A and e_λ $(\lambda \in \mathrm{Sp}\, a \setminus \{0\})$ are the idempotents obtained in Lemma 2.2, then

$$\mathrm{Sp}\,(a - \sum ae_{\lambda_i}) \setminus \{0\} = \mathrm{Sp}\, a \setminus \{\lambda_1, \ldots, \lambda_n, 0\}$$
$$\mathrm{Sp}\,(\sum ae_{\lambda_i}) \setminus \{0\} = \{\lambda_1, \ldots, \lambda_n\}.$$

Proof. Let $a_1 = a - \sum ae_{\lambda_i}$, $a_2 = \sum ae_{\lambda_i}$. Then ${}_{a_1}T|B$ and ${}_{a_2}T|B$ are Riesz operators by Theorem 1.2.8, and it is easily seen that

$$\mathrm{Sp}_B\, a_1 \setminus \{0\} = \mathrm{Sp}_A\, a_1 \setminus \{0\} = \mathrm{Sp}\; {}_{a_1}T|B \setminus \{0\}$$
$$\mathrm{Sp}_B\, a_2 \setminus \{0\} = \mathrm{Sp}_A\, a_2 \setminus \{0\} = \mathrm{Sp}\; {}_{a_2}T|B \setminus \{0\}.$$

By Corollary 2.2.5 and Theorem 1.2.13, $\mathrm{Sp}\; {}_{a_1}T|B \setminus \{0\} = \mathrm{Sp}\; {}_aT|B \setminus \{\lambda_1, \ldots \lambda_n, 0\}$. and hence $\mathrm{Sp}_A\, a_1 \setminus \{0\} = \mathrm{Sp}_A\, a \setminus \{\lambda_1, \ldots, \lambda_n, 0\}$. Similarily $\mathrm{Sp}_A\, a_2 \setminus \{0\} = \{\lambda_1, \ldots, \lambda_n\}$.

Thus, if ${}_aT_a$ is a Riesz operator, we may split up a very much as we can split up a Riesz operator T in $B(X)$. In what we have shown so far it appears that the idempotents e_λ depend on the subalgebra B containing a which is involved in their construction. We now prove that e_λ is independant of B and lies in $A(a)$, the smallest closed subalgebra of A containing a. $A(a)$ is the closure of the set of polynomials in a without constant term.

THEOREM 2.5. e_λ $(\lambda \in \mathrm{Sp}\, a \setminus \{0\})$ is independent of B and lies in $A(a)$.

Proof. Take $\lambda \in \mathrm{Sp}\, a \setminus \{0\}$, and construct idempotents e_λ relative to B and f_λ relative to $A(a)$. Since e_λ commutes with a, e_λ commutes with f_λ. Now $\lambda \notin \mathrm{Sp}(a - af_\lambda)$ but

$$(a - af_\lambda - \lambda 1)^\nu (e_\lambda - e_\lambda f_\lambda) = \{(a - \lambda 1)^\nu (1 - f_\lambda) + (-\lambda)^\nu f_\lambda\}(e_\lambda - e_\lambda f_\lambda)$$
$$= 0.$$

Hence $e_\lambda = e_\lambda f_\lambda$ and similarly we can prove $f_\lambda = e_\lambda f_\lambda$. Thus $e_\lambda = f_\lambda$.

In fact we have proved rather more. Suppose e_λ is an idempotent commuting with a such that for some integer ν $(a - \lambda 1)^\nu e_\lambda = 0$ and $\lambda \notin \mathrm{Sp}(a - ae_\lambda)$. Then the proof of the last theorem shows that e_λ is identical to the idempotent f_λ in $A(a)$. Thus we have the following theorem.

THEOREM 2.6. For each $\lambda \in \mathrm{Sp}\, a \setminus \{0\}$ there exists a unique idempotent e_λ commuting with a such that $\lambda \notin \mathrm{Sp}(a - ae_\lambda)$ and such that, for some integer ν, $(a - \lambda 1)^\nu e_\lambda = 0$. e_λ lies in $A(a)$.

In the particular case when we take A to be the algebra $B(X)$ and a to be a Riesz operator T, then the unique idempotent is the projection onto $\mathrm{Null}(T - \lambda I)^\nu$.

We now proceed to construct a minimal idempotent from the idempotents already constructed.

LEMMA 2.7. If ${}_aT_a$ is a Riesz operator on A and $\nu(a) > 0$, A contains an idempotent e such that eAe is finite-dimensional.

Proof. If $\lambda \in \mathrm{Sp}\, a \setminus \{0\}$, take e to be the idempotent e_λ. Since $(a - \lambda 1)^\nu e = 0$, there exists c in A such that $e = ac = ca$. Hence ${}_eT_e = {}_aT_a\, {}_cT_c = {}_cT_c\, {}_aT_a$ and thus, by Theorem 1.2.8, ${}_eT_e$ is a Riesz operator. Since eAe is a closed subspace of A on which ${}_eT_e$ is the identity operator, it follows that eAe is finite-dimensional by Theorem 1.2.9.

We now need a theorem by Kaplansky, (9) Theorem 3.1. This result is for a general ring, but the proof clearly goes through for a Banach algebra, where a primitive ideal is the kernel of a strictly irreducible representation on a Banach space, rather than on a module. We do not use the full force of the theorem here but we shall need it in a later chapter.

LEMMA 2.8. Let A be a Banach algebra and let B be of the form eAe where e is an idempotent in A. Then there exists a one-to-one correspondence between the primitive ideals of B and those primitive ideals of A not containing B. The mapping is implemented by $P \rightarrow P \cap B$, P primitive in A, and it is a homeomorphism in the topologies of the structure spaces of A and B.

THEOREM 2.9. Let A be a semi-simple Banach algebra which contains an element a for which ${}_aT_a$ is a Riesz operator and $\nu(a) > 0$. Then A contains minimal idempotents.

Proof. By Lemma 2.7 A has a non-zero idempotent e such that eAe is finite-dimensional. Thus eAe contains minimal one-sided ideals, and by Lemma 2.8 eAe is semi-simple. Hence, by Lemma 2.1.5 (12), eAe contains a minimal idempotent, ege say. Since

$$(ege)A(ege) = (ege)(eAe)(ege) = \mathbb{C}\, ege$$

it follows that ege is a minimal idempotent of A.

COROLLARY 2.10. Let A be a semi-simple Banach algebra which contains a non-zero element a such that ${}_aT_a$ is a compact operator. Then A contains minimal idempotents.

Proof. Since A is semi-simple there exists an element b in A such that $\nu(ba) > 0$. Putting $c = ba$ we have ${}_cT_c$ compact, since ${}_cT_c = {}_bT\, {}_aT_a\, T_b$, and the result follows from Theorem 2.9.

We may observe that, by the construction of the minimal idempotent in Theorem 2.9, the minimal idempotent lies in both the right and the left ideals generated by a.

In the next section we study the socle of A; in particular the relationship between finite rank operators on A and the socle of A.

3. The Socle.

We need the following lemmas. The first is well-known.

LEMMA 3.1. If A is a semi-simple Banach algebra with a minimal idempotent e then AeA is a minimal two-sided ideal of A.

Proof. In (12) Lemma 2.8.8 Rickart proves that $\overline{AeA}$ is a minimal closed two-sided ideal, and the same proof shows that AeA is a minimal two-sided ideal of A.

LEMMA 3.2. If A is a semi-simple Banach algebra with minimal idempotents e and f, then ${}_eT_f$ has finite rank.

Proof. By the previous lemma AeA and AfA are minimal two-sided ideals of A. Hence either $AeA \cap AfA = (0)$ or $AeA = AfA$. In the first case, since $eAf \subseteq AeA \cap AfA$, it follows that ${}_eT_f = 0$. In the second case we may write $f = \sum_1^n a_i e b_i$, for some $a_1, \ldots, a_n, b_1, \ldots, b_n$ in A. Then $eAf \subseteq \sum_1^n eAa_i e b_i \subseteq \sum_1^n \mathbb{C} e b_i$. Hence ${}_eT_f$ has finite rank.

NOTE. In fact either ${}_eT_f = 0$ or ${}_eT_f$ has rank one. Probably the easiest way to see this is to observe that $\overline{AeA}$ is a primitive algebra, and hence has a faithful strictly irreducible representation on some Banach space X. The images of e and f will be rank one operators — see the discussion in (12) Chapter II § 4. It is then easy to see that there exist a and b in A such that $f = aeb$.

We now give one of the main theorems of this section.

THEOREM 3.3. Let A be a semi-simple Banach algebra with a socle. Then ${}_aT_c$ has finite rank if a and c lie in the socle of A.

Proof. We may write $a = \sum_1^n a_i e_i$, $c = \sum_1^n c_i e_i$, where $e_1, \ldots, e_n$ are minimal idempotents in A. Then

$$ {}_aT_c = \sum_{i,j=1}^{n} {}_{a_i}T\left({}_{e_i}T_{e_j}\right)T_{c_j} $$

and hence ${}_aT_c$ has finite rank by Lemma 3.2.

COROLLARY 3.4. If A is a semi-simple Banach algebra with a dense socle then ${}_aT_c$ is compact for all a and c in A.

Proof. Given $\varepsilon > 0$, chose a' and c' in the socle of A such that $\|a - a'\| \leq \varepsilon$, and $\|c - c'\| \leq \varepsilon$. Then since

$$ \|abc - a'bc'\| \leq \|ab(c - c')\| + \|(a - a')bc'\| \leq \varepsilon\,(\|a\| + \|c\| + \varepsilon\,)\|b\|, $$

we have

$$ \|{}_aT_c - {}_{a'}T_{c'}\| \leq (\|a\| + \|c\| + \varepsilon)\,\varepsilon. $$

Since ${}_{a'}T_{c'}$ has finite rank it follows that ${}_aT_c$ is compact.

COROLLARY 3.5. A semi-simple Banach algebra A has a socle if and only if, for some non-zero element a in A, ${}_aT_a$ is a compact operator on A.

Proof. Immediate from Theorem 3.3 and Corollary 2.10.

We now have a corollary to Theorem 3.3.

THEOREM 3.6. If A is a semi-simple Banach algebra containing a non-zero element a such that ${}_aT_a$ has finite rank, then the socle of A exists and contains a.

Proof. Since A is semi-simple $Aa \neq (0)$. Let L be a non-zero left ideal of A contained in Aa. Then $L^2 \neq (0)$ since A is semi-simple, and thus there exist elements ba and ca in L with $baca \neq 0$. Then $aca \neq 0$ and $aca \in L \cap aAa$. Hence aAa is a non-zero finite-dimensional subspace of A having non-zero intersection with all non-zero left ideals of A contained in Aa.

Then there exists a non-zero subspace X_1 of aAa together with a left ideal L_1 of A, where $L_1 \subset Aa$ and $L_1 \cap aAa = X_1$, such that for any left ideal L of A with $L \subset Aa$ either $L \cap X_1 = (0)$ or X_1. Let L_2 be the intersection of all left ideals of A with $L \subset Aa$ and $L \cap aAa = X_1$. Then L_2 is a minimal left ideal of A, and so $L_2 = Ae$ for some minimal idempotent e of A. Choose $a_1, \ldots, a_m$ to be a basis of X_1, and extend it to a basis of aAa, say $a_1, \ldots, a_n$. Note that $a_r = a_r e$ $(1 \leq r \leq m)$. If $b \in A$ we may write $aba = \sum_1^n \lambda_r a_r$ and $aeba = \sum_1^n \mu_r a_r$. Then

$$\begin{aligned}(a - ae)b(a - ae) &= aba - abae - aeba + aebae \\ &= \sum_1^n \lambda_r(a_r - a_r e) - \sum_1^n \mu_r(a_r - a_r e) \\ &= \sum_{m+1}^n (\lambda_r - \mu_r)(a_r - a_r e).\end{aligned}$$

Hence the rank of ${}_{(a-ae)}T_{(a-ae)}$ is strictly less than the rank of ${}_aT_a$. Applying this process to $(a - ae)$ in place of a, we see that, after a finite number of repetitions of this process, we obtain an element a' in the socle of A such that ${}_{(a-a')}T_{(a-a')} = 0$. Since A is semi-simple it follows that $a = a'$, and thus a lies in the socle of A.

We observe that, if A is semi-simple with elements a and c such that ${}_aT_c$ is a non-zero finite rank operator, there exists b in A such that $abc \neq 0$ and ${}_{abc}T_{abc}$ has finite rank. Thus the socle of A exists but need not contain a and c. For example, let X be an infinite dimensional Banach space, and let $A = B(X) \oplus B(X) \oplus B(X)$ where $\|(T_1,T_2,T_3)\| = \sup(\|T_1\|,\|T_2\|,\|T_3\|)$. The socle of A consists of all elements (T_1,T_2,T_3) where T_1, T_2, T_3 are finite rank operators. Take $a = (R,I,0)$ and $c = (S,0,I)$, where R and S have finite rank. Then ${}_aT_c$ has finite rank but a and c do not lie in the socle of A. However, we do have the following result when we restrict ourselves to the primitive case.

THEOREM 3.7. If A is a primitive Banach algebra containing non-zero elements a and c such that ${}_aT_c$ has finite rank, then the socle of A exists and contains a and c.

Proof. Since A is primitive and $a \neq 0$, $c \neq 0$, then ${}_aT_c \neq 0$. Let L be a non-zero left ideal of A such that $L \subseteq Ac$. Then $aL \neq (0)$, since A is primitive, and hence $aAc \cap L \neq (0)$. Thus aAc is a non-zero finite-dimensional subspace of A having non-zero intersection with all non-zero left ideals contained in Ac.

As in the proof of Theorem 3.6 we can proceed to find a minimal idempotent e such that $Ae \cap aAc \neq (0)$. Choose $a_1,\dots,a_m$ to be a basis of $Ae \cap aAc$ and extend it to form a basis of aAc, say $a_1,\dots,a_n$. Then $a_re = a_r$ $(1 < r \leqslant m)$. If $b \in A$ we may write $abc = \sum_1^n \lambda_r a_r$. Then

$$ab(c - ce) = \sum_1^n \lambda_r(a_r - a_re) = \sum_{m+1}^n \lambda_r(a_r - a_re).$$

Thus the rank of ${}_aT_{(c\cdot ce)}$ is strictly less than the rank of ${}_aT_c$.

Applying this process to $(c - ce)$ in place of c, we see that, after a finite number of repetitions of this process, we obtain an element c' in the socle of A such that ${}_aT_{(c-c')} = 0$. Since A is primitive it follows that $c = c'$.

By considering right ideals of A contained in aA we can similarily show that a lies in the socle of A.

COROLLARY 3.8. If T_1 and T_2 are non-zero operators in $B(X)$, then ${}_1T_2$ has finite rank if and only if T_1 and T_2 have finite rank.

We may compare this result with those characterising compact and Riesz operators — see Theorems 1.3.3, 1.3.4, 1.3.5.

In § 2 we showed how, in a semi-simple Banach algebra **A**, the existence of a non-zero idempotent e for which ${}_eT_e$ has finite rank implied the existence of a socle in **A**. We may now illustrate more fully the connection between the idempotent e and the minimal idempotents of **A**.

THEOREM 3.9. Let **A** be a semi-simple Banach algebra having a non-zero idempotent e for which ${}_eT_e$ has finite rank. Then there exist minimal idempotents $e_1, \ldots, e_n$ such that $e_ie_j = 0$ $(i \neq j)$ and $e = \sum_1^n e_i$.

Proof. As in the proof of Theorem 3.6 we obtain a minimal idempotent f_1 such that $f_1 \in \mathbf{A}e$ and the rank of ${}_{(e-f_1)}T_{(e-f_1)}$ is strictly less than the rank of ${}_eT_e$. Take $e_1 = ef_1$. Then $ee_1 = e_1 = e_1e$ since $f_1 \in \mathbf{A}e$. Also $e_1{}^2 = ef_1ef_1 = ef_1{}^2 = ef_1 = e_1$, and for b in **A** $e_1be_1 = ef_1bef_1 = \lambda ef_1 = \lambda e_1$ for some $\lambda \in \mathbb{C}$. Hence e_1 is a minimal idempotent which commutes with e. Also $e - e_1$ is an idempotent and ${}_{(e-e_1)}T_{(e-e_1)}$ has rank strictly less than that of ${}_eT_e$, since ${}_{(e-e_1)}T_{(e-e_1)} = {}_eT\left({}_{(e-f_1)}T_{(e-f_1)}\right)T_e$.

Now construct e_2 in relation to $e - e_1$ as e_1 was constructed in relation to e, assuming that $e \neq e_1$. Then e_2 is a minimal idempotent and $(e - e_1)e_2 = e_2 = e_2(e - e_1)$ and hence $e_1e_2 = e_2e_1 = 0$. Continuing in this way we obtain a sequence $\{e_i\}_1^n$ of minimal idempotents such that $e_ie_j = 0$ $(i \neq j)$ and such that, if $f = e - \sum_{1}^{n} e_i$, then ${}_fT_f = 0$. Hence $f = 0$ and $e = \sum_{1}^{n} e_i$.

COROLLARY 3.10. If A is a semi-simple Banach algebra with an element a such that ${}_aT_a$ is a Riesz operator, then for each λ in $\mathrm{Sp}\ a \setminus \{0\}$ the associated idempotent e_λ lies in the socle of A.

4. Riesz elements.

We assume throughout this section that A is semi-simple.

DEFINITION 4.1. An element a of A is called a Riesz element if, for each λ in $\mathrm{Sp}\ a \setminus \{0\}$, there exists an idempotent f_λ in the socle of A that commutes with a and is such that $\lambda \notin \mathrm{Sp}(a - af_\lambda)$.

Note that if $\nu(a) = 0$ then a is a Riesz element.

THEOREM 4.2. If a is a Riesz element of A and $\lambda \in \mathrm{Sp}\ a \setminus \{0\}$, then there exists an idempotent e_λ in the socle of A which commutes with a and is such that $\lambda \notin \mathrm{Sp}(a - ae_\lambda)$ and $(a - \lambda 1)^\nu e_\lambda = 0$ for some integer ν.

Proof. By definition there exists an idempotent f_λ in the socle of A such that $af_\lambda = f_\lambda a$ and $\lambda \notin \mathrm{Sp}(a - af_\lambda)$. Then af_λ is in the socle of A and hence, by Theorem 3.3, ${}_{af_\lambda}T_{af_\lambda}$ has finite rank.
Theorem 2.4, Then, by Lemma 2.2, and Theorem 2.5, there exists an idempotent e_λ in the socle of A and in $A(af_\lambda)$ such that $\lambda \notin \mathrm{Sp}(af_\lambda - af_\lambda e_\lambda)$ and for some ν $(af_\lambda - \lambda 1)^\nu e_\lambda = 0$. Now $ae_\lambda = e_\lambda a$ and $e_\lambda f_\lambda = f_\lambda e_\lambda$ because $e_\lambda \in A(af_\lambda)$. Since $(af_\lambda - \lambda 1)^\nu e_\lambda = 0$, it follows that $f_\lambda e_\lambda = e_\lambda$. Thus $(a - \lambda 1)^\nu e_\lambda = 0$. It remains to prove that $\lambda \notin \mathrm{Sp}(a - ae_\lambda)$. Since $\lambda \notin \mathrm{Sp}(af_\lambda - ae_\lambda)$ and $\lambda \notin \mathrm{Sp}(a - af_\lambda)$, there exist elements b and c in A such that

$$\lambda^{-1} b(af_\lambda - ae_\lambda) = \lambda^{-1}(af_\lambda - ae_\lambda)b = \lambda^{-1}(af_\lambda - ae_\lambda) + b \qquad \text{----(i)}$$

$$\lambda^{-1} c(a - af_\lambda) = \lambda^{-1}(a - af_\lambda)c = \lambda^{-1}(a - af_\lambda) + c. \qquad \text{----(ii)}$$

Multiplying (i) and (ii) by f_λ and e_λ, we have

$$f_\lambda b = bf_\lambda = b,\ e_\lambda b = be_\lambda = 0,\ f_\lambda c = cf_\lambda = 0,\ e_\lambda c = ce_\lambda = 0.$$

Thus

$$\lambda^{-1} b(a - ae_\lambda) = \lambda^{-1}(a - ae_\lambda)b = \lambda^{-1}(af_\lambda - ae_\lambda) + b$$

$$\lambda^{-1} c(a - ae_\lambda) = \lambda^{-1}(a - ae_\lambda)c = \lambda^{-1}(a - af_\lambda) + c.$$

On adding we have

$$\lambda^{-1}(b + c)(a - ae_\lambda) = \lambda^{-1}(a - ae_\lambda)(b + c) = \lambda^{-1}(a - ae_\lambda) + (b + c).$$

Thus $\lambda \notin \mathrm{Sp}(a - ae_\lambda)$ and our proof is complete.

COROLLARY 4.3. An operator T in $B(X)$ is a Riesz operator if and only if, for each λ in $\mathrm{Sp}\, T \setminus \{0\}$, there exists a projection P_λ of finite rank which commutes with T and is such that $\lambda \notin \mathrm{Sp}\, T - TP_\lambda$.

Proof. This is immediate from Theorems 4.2 and 1.2.14.

COROLLARY 4.4. An operator T in $B(X)$ is a Riesz operator if and only if, for each λ in $\mathrm{Sp}\, T \setminus \{0\}$, there exists a closed subspace R_λ with a finite-dimensional subspace N_λ such that $X = N_\lambda \oplus R_\lambda$, N_λ and R_λ are invariant under T, and $T - \lambda I / R_\lambda$ is a homeomorphism.

Proof. If T is a Riesz operator the result is clear. Conversely, there will exist an associated projection P_λ onto N_λ and P_λ will lie in $B(X)$ since R_λ and N_λ are closed. Then $T - \lambda I / R_\lambda$ being a homeomorphism will imply $\lambda \notin \mathrm{Sp}\, T - TP_\lambda$. The result then follows from Corollary 4.3.

NOTE. T is a Riesz operator on X if and only if T is a Riesz element of $B(X)$.

LEMMA 4.5. Let a be a Riesz element of A, and let $\{e_\lambda\}$ $(\lambda \in \mathrm{Sp}\ a \setminus \{0\})$ be the set of idempotents produced in the last theorem. Then

(i) $e_\lambda e_\mu = 0 \quad (\lambda \neq \mu)$

(ii) $\mathrm{Sp}\ (a - \sum_1^n ae_{\lambda_i}) \setminus \{0\} = \mathrm{Sp}\ a \setminus \{\lambda_1, \ldots, \lambda_n, 0\}$.

Proof. If $\lambda, \mu \in \mathrm{Sp}\ a \setminus \{0\}$ and $\lambda \neq \mu$ then the complex polynomials $(t - \lambda 1)^{\nu_\lambda}$ and $(t - \mu 1)^{\nu_\mu}$ are relatively prime, and hence there exist polynomials $p(t)$ and $q(t)$ such that

$$p(t)\ (t - \lambda 1)^{\nu_\lambda} + q(t)\ (t - \mu 1)^{\nu_\mu} = 1.$$

Hence

$$p(a)\ (a - \lambda 1)^{\nu_\lambda} e_\lambda e_\mu + q(a)\ (a - \mu 1)^{\nu_\mu} e_\lambda e_\mu = e_\lambda e_\mu.$$

Since e_λ commutes with a it follows that $e_\lambda e_\mu = 0$.

(ii) Denote $\sum_1^n e_{\lambda_i}$ by v. Then by (i) v is an idempotent. If $\lambda \notin \mathrm{Sp}\ a$ and $\lambda \neq 0$, there exists c in A such that

$$\lambda^{-1} ac = \lambda^{-1} a + c.$$

Then $\lambda^{-1}(a - av)(c - vc) = \lambda^{-1}(1 - v)ac = \lambda^{-1}(a - av) + (c - vc)$. Thus $\lambda^{-1}(a - av)$ has a right quasi-inverse, and similarily it has a left quasi-inverse. Thus $\lambda \notin \mathrm{Sp}\ (a - av)$.

Since $\lambda_i \notin \mathrm{Sp}\ (a - ae_{\lambda_i})$ there exists c in A such that

$$\lambda_i^{-1}(a - ae_{\lambda_i})c = \lambda_i^{-1}(a - ae_{\lambda_i}) + c.$$

Now $(a - ae_{\lambda_i})(1 - \sum_{j \neq i} e_{\lambda_j}) = a - av$ by (i), and thus, if $d = (1 - \sum_{j \neq i} e_{\lambda_j})c$

$$\lambda_i^{-1}(a - av)d = \lambda_i^{-1}(a - av) + d.$$

Thus as before we have $\lambda_i \notin \mathrm{Sp}\ (a - av) \quad (1 \leq i \leq n)$.

If $\lambda \in \mathrm{Sp}\ a \setminus \{\lambda_1, \ldots, \lambda_n, 0\}$ then, for some integer ν, $(a - \lambda 1)^\nu e_\lambda = 0$. But by (i)

$$(a - av - \lambda 1)^\nu e_\lambda = (a - \lambda 1)^\nu (1 - v)e_\lambda + (-\lambda)^\nu ve_\lambda = 0,$$

and hence $\lambda \in \mathrm{Sp}\ (a - av)$.

LEMMA 4.6. If a is a Riesz element of A, then Sp a is countable, with 0 as the only possible accumulation point.

Proof. Take $\lambda \in \mathrm{Sp}\, a \setminus \{0\}$. Then $\lambda \notin \mathrm{Sp}\,(a - ae_\lambda)$ and hence there exists a neighbourhood U of λ such that, if $\mu \in U$, then $\mu \notin \mathrm{Sp}\,(a - ae_\lambda)$. If $\mu \in U \setminus \{\lambda\}$, then for some c in A

$$(a - ae_\lambda - \mu 1)(c + 1) = -\mu 1.$$

Multiplying on the left by e_λ we have $e_\lambda c = 0$, and thus

$$(a - \mu 1)(c + 1) = -\mu 1 + ae_\lambda .$$

Since $\mu \neq \lambda$ there exists a complex polynomial $p(t)$ such that

$$p(t) = \frac{-t}{(\lambda - \mu)^\nu} \frac{(t - \lambda)^\nu + (\lambda - \mu)^\nu}{(t - \lambda) + (\lambda - \mu)} = \frac{-t}{(\lambda - \mu)^\nu} \frac{(t - \lambda)^\nu + (\lambda - \mu)^\nu}{t - \mu}$$

Then we have

$$(a - \mu 1)(p(a) e_\lambda + c + 1) = \left(-1/(\lambda - \mu)^\nu\right)(a(a - \lambda 1)^\nu e_\lambda) - ae_\lambda - \mu 1 + ae_\lambda = -\mu 1.$$

Thus $a - \mu 1$ has a right inverse and similarily has a left inverse. Hence $\mu \notin \mathrm{Sp}\, a$ and so λ is an isolated point of Sp a. Since Sp a is compact, it follows that Sp a is countable, with 0 as the only possible accumulation point.

We now give the main result of this section.

THEOREM 4.7. If A is semi-simple with Riesz elements a and c, then ${}_aT_c$ is a Riesz operator on A.

Proof. By Lemmas 4.5 and 4.6 we have that, given $\varepsilon > 0$, there exist idempotents $e_{\lambda_1}, \dots, e_{\lambda_n}$ and $f_{\mu_1}, \dots, f_{\mu_n}$ in the socle of A which commute with a and c respectively, and for which $\nu(a - \sum ae_{\lambda_i}) < \varepsilon$ and $\nu(c - \sum cf_{\mu_i}) < \varepsilon$. For b in A we have

$abc = (a - \sum ae_{\lambda_i})bc + (\sum ae_{\lambda_i})b(c - \sum cf_{\mu_j}) + \sum \sum ae_{\lambda_i} bcf_{\mu_j}$.

Since a commutes with $a - \sum ae_{\lambda_i}$, $\nu(\sum ae_{\lambda_i}) < \nu(a) + \varepsilon$, and by Theorem 3.3 ${}_{ae_{\lambda_i}}T_{cf_{\mu_j}}$ has finite rank. Thus, again by commutativity,

${}_aT_c$ is the sum of a finite rank operator and an operator whose spectral radius is less than $(\nu(a) + \nu(c) + \varepsilon)\varepsilon$. Hence by the Ruston characterisation ${}_aT_c$ is a Riesz operator.

COROLLARY 4.8. If A is semi-simple, a is a Riesz element of A if and only if ${}_aT_a$ is a Riesz operator on A.

Proof. If $\nu(a) = 0$ then $\nu({}_aT_a) = \nu(a) = 0$ and the result is trivial. If $\nu(a) > 0$ and ${}_aT_a$ is a Riesz operator, then by Theorem 2.4 and Corollary 3.10 a is a Riesz element. The converse follows immediately from the previous theorem.

To obtain a full generalisation of the concept of Riesz operators to a general Banach algebra A we should like to be able to show that, if A is primitive and $\nu(a) > 0$ and $\nu(c) > 0$, ${}_aT_c$ a Riesz operator implies that both a and c are Riesz elements. A major difficulty seems to be the fact that $\nu({}_aT_c)$ does not appear to be necessarily equal to $\nu(a)\,\nu(c)$.

5. The Spectral Properties of the Operator ${}_aT_c$.

We start with a fairly general theorem.

THEOREM 5.1. If λ is a non-zero eigen value of ${}_aT_c$ such that $\mathrm{Null}({}_aT_c - \lambda I)$ is finite-dimensional, then there exists $\mu \in \mathrm{Sp}\ c \setminus \{0\}$ and a non-zero element d of A such that $\lambda/\mu \in \mathrm{Sp}\ a$ and $ad = \lambda/\mu\, d$ and $dc = \mu d$.

Proof. Take a non-zero element b in $\mathrm{Null}({}_aT_c - \lambda I)$. Then $bc^n \in \mathrm{Null}({}_aT_c - \lambda I)$ for all positive integers n. Thus, since

$\mathrm{Null}({}_aT_c - \lambda I)$ is finite-dimensional, there exists a polynomial q such that $b.q(c) = 0$. Now $bc^n \neq 0$, since $a^n bc^n = \lambda^n b$, and hence we may write

$$bc^n \prod_1^m (c - \mu_i 1) = 0$$

for some non-zero factors μ_i. Hence there exists a polynomial p and μ in $\mathrm{Sp}\, c \setminus \{0\}$ such that, if $d = b.p(c)$, $d \neq 0$ and $dc = \mu d$. But $d \in \mathrm{Null}({}_aT_c - \lambda I)$ and hence $ad = {}^1\!/\!_\mu\, adc = {}^\lambda\!/\!_\mu\, d$.

COROLLARY 5.2. If ${}_aT_c$ is a Riesz operator and λ is a non-zero eigen value of ${}_aT_c$, there exists $\mu \in \mathrm{Sp}\, c \setminus \{0\}$ such that $\lambda/\mu \in \mathrm{Sp}\, a \setminus \{0\}$.

It is natural to ask whether we can obtain a converse to this corollary i.e. if ${}_aT_c$ is a Riesz operator and $\mu \in \mathrm{Sp}\, c \setminus \{0\}$ and $\lambda \in \mathrm{Sp}\, a \setminus \{0\}$, is $\lambda\mu \in \mathrm{Sp}\, {}_aT_c$? It is clear from the following example that we must restrict ourselves to the primitive case.

EXAMPLE 5.3. Take A to be $B(X) \oplus B(X) \oplus B(X)$ for some Banach space X. Let $a = (x \otimes y, 0, \lambda I)$, $c = (x \otimes y, \mu I, 0)$ where $(x,y) = 1$. If $b = (R,S,T)$, then ${}_aT_c(b) = \big((Rx,y)\, x \otimes y, 0, 0\big)$. Thus ${}_aT_c$ has finite rank and so is a Riesz operator. 1 is clearly the only non-zero eigen value but $\lambda \notin \mathrm{Sp}\, a$ and $\mu \notin \mathrm{Sp}\, c$. Indeed these are eigen values in the sense that there exist non-zero elements a' and c' such that $aa' = \lambda a'$ and $c'c = \mu c'$.

If A is primitive and λ and μ are eigen values of a and c respectively in the above sense, then $\lambda\mu$ is an eigen value of ${}_aT_c$. For suppose that a' and c' are non-zero elements of A such that $aa' = \lambda a'$ and $c'c = \mu c'$. Since A is primitive we may find b in A such that $a'bc' \neq 0$. Then ${}_aT_c(a'bc') = \lambda\mu\, a'bc'$. It is not clear, however, whether ${}_aT_c$ being a Riesz operator will imply that each element in the spectra of a and c is an eigen value in the above sense.

Theorem 5.1 allows us to demonstrate the existence of minimal idempotents under the assumption that ${}_aT_c$ is a Riesz operator for some elements a and c. Previously we have assumed that ${}_aT_a$ was a Riesz operator for some element a — see Theorem 2.9.

THEOREM 5.4. If A is semi-simple and ${}_aT_c$ is a Riesz operator with non-zero spectral radius, then A contains minimal idempotents.

Proof. If $\lambda \in \mathrm{Sp}\ {}_aT_c \setminus \{0\}$, then, by Theorem 5.1, there exists μ in $\mathrm{Sp}\ c \setminus \{0\}$ and a non-zero element d in A such that

$$ad = \lambda/\mu\ d, \quad dc = \mu d.$$

Now ${}_aT_c$ restricted to $\overline{dAd}$ is λI, where I is the identity operator, and is a Riesz operator by Theorem 1.2.11. Hence $\overline{dAd}$ is finite-dimensional, and thus so is dAd. Then, by Theorem 3.6, the socle of A exists and contains d.

If ${}_aT_c$ is a compact operator we may reduce our conditions a little.

THEOREM 5.5. If A is semi-simple and ${}_aT_c$ is a non-zero compact operator, then A contains minimal idempotents.

Proof. Since ${}_aT_c \neq 0$, there exists b in A such that $abc \neq 0$. Putting $d = abc$ we have ${}_dT_d$ is compact, since ${}_dT_d = {}_aT_c \cdot {}_{bc}T_{ab}$. Hence by Corollary 2.10 A contains minimal idempotents.

We are unable to obtain any further information about the spectrum of ${}_aT_c$ only on the assumption that ${}_aT_c$ is a Riesz operator. Thus we now assume that A is semi-simple and that a and c are Riesz elements of A. By Theorem 4.7 ${}_aT_c$ is a Riesz operator.

THEOREM 5.6. $\lambda \in \mathrm{Sp}\ {}_aT_c \setminus \{0\}$ if and only if there exists μ in $\mathrm{Sp}\ c \setminus \{0\}$ such that $\lambda/\mu \in \mathrm{Sp}\ a \setminus \{0\}$, and $\mathrm{Null}({}_aT - \frac{\lambda}{\mu}I) \cap \mathrm{Null}(T_c - \mu I)$ is not (0).

Proof. Theorem 5.1 gives the implication in one direction. Conversely, if $\lambda/\mu \in \mathrm{Sp}\ a \setminus \{0\}$ and $\mu \in \mathrm{Sp}\ c \setminus \{0\}$, suppose that d lies in $\mathrm{Null}({}_aT - \frac{\lambda}{\mu}I)$ and in $\mathrm{Null}(T_c - \mu I)$. Then ${}_aT_c(d) = \lambda\, abc$.

COROLLARY 5.7. If A is primitive and a and c are Riesz elements, then $\lambda \in \mathrm{Sp}\ {}_aT_c \setminus \{0\}$ if and only if there exists μ in $\mathrm{Sp}\ c \setminus \{0\}$ such that $\lambda/\mu \in \mathrm{Sp}\ a \setminus \{0\}$.

Proof. If $\lambda/\mu \in \mathrm{Sp}\ a \setminus \{0\}$ and $\mu \in \mathrm{Sp}\ c \setminus \{0\}$, then $\mathrm{Null}({}_aT - \frac{\lambda}{\mu}I)$ and $\mathrm{Null}(T_c - \mu I)$ are non-zero right and left ideals respectively, and they have non-zero intersection since A is primitive.

For $\lambda \in \mathrm{Sp}\ {}_aT_c \setminus \{0\}$, let $A = N_\lambda \oplus R_\lambda$ be the corresponding decomposition of A under the Riesz operator ${}_aT_c$, and let P_λ be the projection onto N_λ. Let $\{\mu_1, \ldots, \mu_n\} = \{\mu \in \mathrm{Sp}\ c \setminus \{0\};\ \lambda/\mu \in \mathrm{Sp}\ a\}$. Let $\lambda_i = \lambda/\mu_i$ $(1 \leq i \leq n)$. Let e_{λ_i} be the idempotent associated with a and λ_i, and let f_{μ_i} be the idempotent associated with c and μ_i — see Theorem 2.6. We now determine P_λ. Our proofs go through very much as in § 4, Chapter I.

LEMMA 5.8. (i) If $b \in N_\lambda$ and $bf_{\mu_i} = 0$ $(1 \leq i \leq n)$ then $b = 0$.
(ii) If $b \in N_\lambda$, then $b = \sum bf_{\mu_i}$ and $b = \sum e_{\lambda_i} b$.

Proof. (i) Let $d = c - \sum f_{\mu_i} c$. Then $\mathrm{Sp}\ d \setminus \{0\} = \mathrm{Sp}\ c \setminus \{\mu_1, \ldots, \mu_n, 0\}$, by Lemma 4.5. ${}_aT_d$ is a Riesz operator since c commutes with f_{μ_i}, and $\lambda \notin \mathrm{Sp}\ {}_aT_d$ by Corollary 5.2. But $({}_aT_d - \lambda I)\check{}\ b = 0$ since $bf_{\mu_i} = 0$, and thus $b = 0$.

(ii) If $b \in M_\lambda$, then $b - \sum bf_{\mu_i}$ is contained in N_λ since each f_{μ_i} commutes with c. Since $(b - \sum bf_{\mu_i})\, f_{\mu_j} = 0 \quad (1 \leq j \leq n)$, it follows from (i) that $b = \sum bf_{\mu_i}$.

Similarly $b = \sum e_{\lambda_i} b$.

THEOREM 5.9. $P_\lambda = \sum {}_{e_{\lambda_i}}T_{f_{\mu_i}}$.

Proof. Suppose $b \in N_\lambda$. Then $b = \sum bf_{\mu_i}$. We first prove that $bf_{\mu_i} = e_{\lambda_i} bf_{\mu_i}$. Let $d = cf_{\mu_i}$. Since c and f_{μ_i} commute ${}_aT_d$ is a Riesz operator, and by Theorem 2.4 $\mathrm{Sp}\, d \setminus \{0\} = \{\mu_i\}$. Now

$$({}_aT_d - \lambda I)^\nu (bf_{\mu_i}) = \{({}_aT_c - \lambda I)^\nu b\} f_{\mu_i} = 0.$$

Thus, applying Lemma 5.8 to the operator ${}_aT_d$, we have

$$e_{\lambda_i} bf_{\mu_i} = bf_{\mu_i}.$$

Hence, if $b \in N_\lambda$, $b = \sum e_{\lambda_i} bf_{\mu_i}$.

To prove the converse we choose some fixed μ_i, call it μ, and prove that all elements of the form $e_\rho bf_\mu$ lie in N_λ, where $\rho = \lambda/\mu$. Let $e_\rho Af_\mu = V$. Since e_ρ and f_μ lie in the socle of A, V is finite-dimensional by Theorem 3.3. ${}_aT_c$ maps V into itself since a commutes with e_ρ and c commutes with f_μ. Exactly as in the proof of Theorem 1.4.2 we prove that λ is the only non-zero eigen value of ${}_aT_c | V$, using the fact that, if $b \in N_{\lambda'} \cap V$, then $b = \sum e_{\lambda'/\mu_i'} bf_{\mu_i'}$ and $b = e_\rho bf_\mu$, where $\mu_1', \ldots, \mu_n' \in \mathrm{Sp}\, c \setminus \{0\}$ and $\lambda'/\mu_1', \ldots, \lambda'/\mu_n' \in \mathrm{Sp}\, a \setminus \{0\}$. Hence by orthogonality $b = 0$. In addition, 0 is not an eigen value of ${}_aT_c | V$ since, if $ae_\rho bf_\mu c = 0$, then $e_\rho bf_\mu c = 0$ because $(a - \rho 1)^\nu e_\rho bf_\mu c = 0$. But $e_\rho bf_\mu (c - \mu 1)^\nu = 0$, and hence $e_\rho bf_\mu = 0$. Thus λ is the only eigen value of ${}_aT_c | V$, and hence for some integer ν we have

$$({}_aT_c - \lambda I)^\nu V = (0).$$

Hence $b \in N_\lambda$ if and only if $b = \sum e_{\lambda_i} bf_{\mu_i}$. If $b \in R_\lambda$, let $b = ({}_aT_c - \lambda I)^\nu d$. Then $\sum e_{\lambda_i} bf_{\mu_i} = ({}_aT_c - \lambda I)^\nu (\sum e_{\lambda_i} df_{\mu_i}) = 0$ as $\sum e_{\lambda_i} df_{\mu_i} \in N_\lambda$. Thus $P_\lambda = \sum {}_{e_{\lambda_i}}T_{f_{\mu_i}}$.

IV REPRESENTATION THEORY

1. Introduction.

In previous chapters we have considered Riesz operators on a Banach space and Riesz elements of a general Banach algebra. We now show how, in a primitive Banach algebra, we may represent a Riesz element by a Riesz operator, and obtain a sort of converse. We also show how the spectral properties of a Riesz element are closely related to those of its image under the representation.

We start, however, by describing the concept of a dual representation which was introduced by Bonsall and Duncan in (4).

2. Dual representations.

We start with the definition of a pair of Banach spaces in normed duality — see (12) Definition 2.4.8.

DEFINITION 2.1. Two (real or complex) Banach spaces X and Y are said to be in normed duality provided there exists a function defined on X x Y to the field of scalars, whose value at the pair x, y is denoted by (x,y), which satisfies the following conditions

(i) $(x,y) = 0$ for every y implies $x = 0$

$(x,y) = 0$ for every x implies $y = 0$

(ii) (x,y) is linear in x for fixed y and in y for fixed x

(iii) there exists a positive constant β such that

$$|(x,y)| \leq \beta \|x\| \|y\|, \qquad x \in X,\ y \in Y.$$

We need the following definition from (12), Definition 2.4.10.

DEFINITION 2.2. Given Banach spaces X, Y in normed duality with respect to (,), the operators T in B(X) and S in B(Y) are said to be adjoint with respect to (,) if

$$(Tx,y) = (x,Sy) \qquad (x \in X,\ y \in Y).$$

There exists at most one such S for a given T by condition (i) and we denote this unique S (if it exists) by T^*, and call it the adjoint of T. Similarly we denote the unique adjoint of S (if it exists) by S^*. It is clear that if $T_1, T_2 \in B(X)$ and have adjoints on Y with respect to (,), then so do $T_1 + T_2$, λT_1, T_1T_2 and $(T_1 + T_2)^* = T_1^* + T_2^*$, $(\lambda T_1)^* = \lambda T_1^*$, $(T_1T_2)^* = T_2^*T_1^*$. If T has an adjoint T^*, then T^* has an adjoint $(T^*)^*$ and $(T^*)^* = T$.

Denote by $B(X,Y,(\ , \ ))$ the algebra of all operators in B(X) with an adjoint in B(Y). The following result is in (4), Proposition 7.

THEOREM 2.3. $B(X,Y,(\ , \ ))$ is a Banach algebra under the norm $\|T\|' = \max(\|T\|, \|T^*\|)$.

Proof. We only have to show that the norm is complete. If $\{T_n\}$ is a Cauchy sequence in $B(X,Y,(\ , \ ))$ there exists T in B(X) and S in B(Y) such that $\|T_n - T\| \to 0$ and $\|T_n^* - S\| \to 0$. Since

$$(Tx,y) = \lim (T_nx,y) = \lim (x,T_n^*y) = (x,Sy) \qquad (x \in X,\ y \in Y),$$

it follows that $S = T^*$. Then $\|T_n - T\|' \to 0$.

For $x \in X$, $y \in Y$, denote by $x \otimes y$ the bounded linear operator on X given by $x \otimes y\,(u) = (u,y)\,x \quad (u \in X)$. We denote by $F(X,Y,(\ , \ ))$ the subalgebra of all finite rank operators in $B(X,Y,(\ , \ ))$. In (12) page 64 it is shown that $T \in F(X,Y,(\ , \ ))$ if and only if there exist $x_1,\dots,x_n$ in X and $y_1,\dots,y_n$ in Y such that $T = \sum x_i \otimes y_i$.

We now define a dual representation — see (4).

DEFINITION 2.4. A dual representation of a Banach algebra A on $(X,Y,(\ ,\ ))$ is a mapping $a \longrightarrow S_a$ of A into $B(X,Y,(\ ,\ ))$ such that $a \longrightarrow S_a$ is a representation of A on X.

It is easily seen that in this case $a \longrightarrow S_a^*$ is an anti-representation of A on Y whose kernel is also the kernel of the representation $a \longrightarrow S_a$.

A dual representation is said to be dually strictly irreducible if $a \longrightarrow S_a$ and $a \longrightarrow S_a^*$ are both strictly irreducible.

We shall use dual representations in connection with a primitive Banach algebra having minimal one-sided ideals. The existence of such representations is shown by Rickart in (12) Chapter II, §4, and we start the next section by stating, and sketching the proof of, his main result.

3. Representation Theory.

We take A to be a primitive Banach algebra with minimal idempotents. The existence of minimal idempotents is ensured if, for some elements a and c of A, ${}_aT_c$ is a Riesz operator with non-zero spectral radius — see Theorem 3.5.4.

THEOREM 3.1. There exist Banach spaces X and Y in normed duality and a norm-decreasing isomorphism $a \longrightarrow S_a$ of A into $B(X,Y,(\ ,\ ))$ such that the socle of A is mapped onto $F(X,Y,(\ ,\ ))$.

Proof. Let e be a minimal idempotent in A. We construct Banach spaces X and Y in the usual way. We take $X = Ae$, $Y = eA$ with the norm induced by A. Since $eAe = \mathbb{C}e$, we may define a bilinear form (x,y) by $(x,y)\,e = yx \quad (x \in X,\ y \in Y)$. Then it is easily seen that

X and Y are in normed duality. For a in A we define the operator $S_a \in B(X)$ by $S_a x = ax$ $(x \in X)$. Since $(S_a x, y)e = yax$, $S_a{}^*$ exists and $S_a{}^* y = ya$ $(y \in Y)$. Since $S_a(X) = aAe$ and A is primitive, it follows that $S_a = 0$ implies $a = 0$. In addition, $\|S_a\| \leq \|a\|$ and $\|S_a{}^*\| \leq \|a\|$, and thus $a \to S_a$ is a norm-decreasing isomorphism of A into $B(X,Y,(\ ,\ ))$. If $x \in X$, $y \in Y$, then $S_{xy} = x \otimes y$ and xy lies in the socle of A. Conversely, if a lies in the socle of A, then $S_a(X) = aAe$ is finite-dimensional by Theorem 3.3.3, and thus S_a has finite rank. Hence the socle of A maps onto $F(X,Y,(\ ,\ ))$.

THEOREM 3.2. Let $b \to S_b$ be the dual representation constructed in the proof of Theorem 3.1. (i) If a and c are non-zero elements of A such that ${}_aT_c$ is compact, then S_a and $S_c{}^*$ are compact.

(ii) If a and c are elements of A such that ${}_aT_c$ is a Riesz operator with non-zero spectral radius, then S_a and $S_c{}^*$ are Riesz operators.

Proof. (i) Since $c \neq 0$ there exist elements x in X and y in Y such that $(S_c x, y) = 1$. Hence $(e \otimes y)S_c(x \otimes e) = e \otimes e$, and thus, since the image of the socle of A is $F(X,Y,(\ ,\ ))$, it follows that there exist elements b and d in A such that $e = bcd$. Then ${}_aT_e$ is compact since ${}_aT_e = T_d({}_aT_c)T_b$. Now ${}_aT_e | Ae = S_a$, and hence S_a is compact. Similarily we prove $S_c{}^*$ compact.

(ii) In the proof of Theorem 3.5.1 we showed that, if $\lambda \in \mathrm{Sp}\ {}_aT_c \setminus \{0\}$, there exists μ in $\mathrm{Sp}\ c \setminus \{0\}$ and a non-zero element d of A such that

$$ad = \lambda/\mu\, d, \quad dc = \mu\, d.$$

It followed that dAd is finite-dimensional and thus, by the construction used in Theorem 3.3.6, there exists a minimal idempotent f in Ad. Then $fc = \mu f$, and thus ${}_aT_c$ maps Af into Af and ${}_aT_c | Af = \mu\, {}_aT_f | Af$. Thus ${}_aT_f | Af$ is a Riesz operator and we now proceed to relate it to S_a. If $S_f = x_1 \otimes y_1$, then $(x_1, y_1) = 1$ and we may map Af isomorphically onto X by $b \longrightarrow S_b x_1$. Since $\|S_b\| \leq \|b\|$, this is a homeomorphism by the closed graph theorem. In the corresponding homeomorphism between $B(Af)$ and $B(X)$ ${}_aT_f|Af$ corresponds to S_a. Thus S_a is a Riesz operator, and similarly we may prove that $S_c{}^*$ is a Riesz operator.

NOTE. The hypotheses in Theorem 3.2 will imply the existence of minimal idempotents in A, and thus the representation of A will exist.

If ${}_aT_a$ is a Riesz operator we can relate the decomposition of a with the decomposition of S_a. Let $\{e_\lambda ; \lambda \in \mathrm{Sp}\ a \setminus \{0\}\}$ be the idempotents associated with a.

THEOREM 3.3. If ${}_aT_a$ is a Riesz operator, then S_a is a Riesz operator and $\mathrm{Sp}\ a \setminus \{0\} = \mathrm{Sp}\ S_a \setminus \{0\}$. If $\lambda \in \mathrm{Sp}\ a \setminus \{0\}$ and P_λ is the decomposition projection on X associated with S_a, then $S_{e_\lambda} = P_\lambda$.

Proof. We observe that $\nu(S_a) \leq \nu(a)$. Hence if $\nu(a) = 0$, S_a is quasi-nilpotent and hence a Riesz operator. If $\nu(a) > 0$ then $\nu({}_aT_a) = (\nu(a))^2 > 0$, and thus by Theorem 3.2 S_a is a Riesz operator.

Clearly $\mathrm{Sp}\ a \setminus \{0\} \supseteq \mathrm{Sp}\ S_a \setminus \{0\}$. If $\lambda \in \mathrm{Sp}\ a \setminus \{0\}$, then $(a - \lambda 1)^\nu e_\lambda = 0$ and so $(S_a - \lambda I)^\nu S_{e_\lambda} = 0$. Hence $\lambda \in \mathrm{Sp}\ S_a \setminus \{0\}$ since $S_{e_\lambda} \neq 0$. Thus $\mathrm{Sp}\ a \setminus \{0\} = \mathrm{Sp}\ S_a \setminus \{0\}$.

If $\lambda \in \mathrm{Sp}\ a \setminus \{0\}$, $\lambda \notin \mathrm{Sp}\ (a - ae_\lambda)$, and hence, if $(S_a - S_a . S_{e_\lambda} - \lambda I)^\nu x = 0$, then $x = 0$. Now

$$(S_a - S_a . S_{e_\lambda} - \lambda I)^\nu = (S_a - \lambda I)^\nu (I - S_{e_\lambda}) + (-\lambda)^\nu S_{e_\lambda}.$$

Hence, if $x \in \mathrm{Null}(S_a - \lambda I)^\nu$, $(I - S_{e_\lambda})x \in \mathrm{Null}(S_a - S_a . S_{e_\lambda} - \lambda I)^\nu$ and so $S_{e_\lambda} x = x$. If $x \in \mathrm{Range}(S_a - \lambda I)^\nu$, then $S_{e_\lambda} x = 0$ since $(a - \lambda 1)^\nu e_\lambda = 0 = e_\lambda (a - \lambda 1)^\nu$. Thus $P_\lambda = S_{e_\lambda}$.

NOTE. As before, possible confusion over the index ν does not arise since we may choose ν to be greater than the index of λ for a and the index of λ for S_a.

We now give a type of converse to Theorem 3.2.

THEOREM 3.4. Let A be a primitive Banach algebra having a continuous isomorphism $a \longrightarrow S_a$ into $B(X,Y,(\;,\;))$ such that the image of A contains $F(X,Y,(\;,\;))$. Then

(i) if S_a (S_a^*) is a compact operator there exists a non-zero element c such that ${}_aT_c$ $({}_cT_a)$ is a compact operator,

(ii) if S_a (S_a^*) is a Riesz operator there exists an element c such that ${}_aT_c$ $({}_cT_a)$ is a Riesz operator with $\nu({}_aT_c) = \nu(S_a)$ $\left(\nu({}_cT_a) = \nu(S_a^*)\right)$.

Proof. It is clear from the fact that the image of A contains $F(X,Y,(\;,\;))$ that A contains minimal idempotents. Let c be a minimal idempotent. Then S_c is of the form $x \otimes y$, where $(x,y) = 1$, and thus Ac is homeomorphic to X while cA is homeomorphic to Y — see (12) Lemma 2.4.13. Since ${}_aT_c\,(bc) = abc^2 = abc$, it follows that, in the homeomorphism between B(Ac) and B(X), ${}_aT_c|Ac$ corresponds to S_a.

(i) If S_a is compact on X then ${}_aT_c|Ac$ is compact on Ac. But ${}_aT_c = ({}_aT_c|Ac) \circ T_c$, and thus ${}_aT_c$ is compact.

Similarly, if S_a^* is compact, then ${}_cT_a$ is compact.

(ii) In the same way, if S_a is a Riesz operator, ${}_aT_c|Ac$ is a Riesz operator. We now prove that $\mathrm{Sp}\ {}_aT_c \setminus \{0\} = \mathrm{Sp}\ ({}_aT_c|Ac) \setminus \{0\}$. Suppose $\lambda \notin \mathrm{Sp}\ ({}_aT_c|Ac)$, $\lambda \neq 0$, and let S be the inverse of ${}_aT_c|Ac - \lambda I$. Extend S to the whole of A by defining $S(b - bc) = -\frac{1}{\lambda}(b - bc)$. Then S is linear on A and it is easily seen that S is bounded on A. $S({}_aT_c - \lambda I)b = S(abc) - \lambda S(b) = S(abc) - \lambda S(bc) + b - bc = bc + b - bc = b$.

Similarly $({}_aT_c - \lambda I)S = I$, and thus $\lambda \notin \operatorname{Sp} {}_aT_c$. If $\lambda \in \operatorname{Sp} {}_aT_c | Ac \setminus \{0\}$, then, for some non-zero element b in Ac, ${}_aT_c\, b = \lambda b$. Thus $\operatorname{Sp} {}_aT_c \setminus \{0\} = \operatorname{Sp} ({}_aT_c | Ac) \setminus \{0\} = \operatorname{Sp} S_a \setminus \{0\}$, and in particular $\nu({}_aT_c) = \nu(S_a)$.

We now prove that ${}_aT_c$ is a Riesz operator. If $\lambda \in \operatorname{Sp} {}_aT_c \setminus \{0\}$ then $\lambda \in \operatorname{Sp} ({}_aT_c | Ac) \setminus \{0\}$. If $({}_aT_c - \lambda I)^{\vee} b = 0$ then clearly $b \in Ac$, and hence $\operatorname{Null} ({}_aT_c - \lambda I)^{\vee} = \operatorname{Null} ({}_aT_c | Ac - \lambda I)^{\vee}$. Also $({}_aT_c - \lambda I)(b - bc) = -\lambda(b - bc)$, and thus

$$\operatorname{Range} ({}_aT_c - \lambda I)^{\vee} = A(1 - c) \oplus \operatorname{Range} ({}_aT_c | Ac - \lambda I)^{\vee} .$$

If $b \in \operatorname{Range} ({}_aT_c - \lambda I)^{\vee}$ and $b = b_1 + b_2$ where $b_1 \in \operatorname{Range} ({}_aT_c | Ac - \lambda I)^{\vee}$ and $b_2 \in A(1 - c)$, then $b_1 = bc$ and $b_2 = b(1 - c)$. Thus it is clear that $\operatorname{Range} ({}_aT_c - \lambda I)^{\vee}$ is closed, and hence ${}_aT_c$ is a Riesz operator.

Similarly, if $S_a{}^*$ is a Riesz operator, ${}_cT_a$ is a Riesz operator.

NOTE. The condition in (ii) that $\nu({}_aT_c) = \nu(S_a)$ prevents the result from being trivial, for otherwise we could take any c in A for which $\nu(c) = 0$ and then $\nu({}_aT_c) = 0$, since $\nu({}_aT_c) \leq \nu(a)\, \nu(c) = 0$.

V COMPACT BANACH ALGEBRAS

1. Introduction.

There are two different approaches to the study of Riesz or compact operators. One may study a single operator or one may study the class of all Riesz, or compact, operators on a Banach space. In the same two ways one may study a general Banach algebra, and so far we have essentially adopted the first approach. We now choose the second. Since the class of Riesz operators on a Banach space is not closed under the usual operations of addition, multiplication, etc. we restrict ourselves to the compact case. We study a Banach algebra A for which ${}_aT_a$ is a compact operator for all elements a in A. We call such an algebra a compact algebra, and an example of such an algebra is the algebra of all compact operators on a Banach space.

In § 2 the main results are a representation theorem for a primitive compact Banach algebra and a structure theorem, which shows that the structure space of a compact Banach algebra is discrete under the hull-kernel topology. This means that a compact Banach algebra has as rich a structure as a completely continuous Banach algebra, although the class of compact Banach algebras is very much wider than the class of completely continuous Banach algebras.

In § 3 we impose an additional restriction on a primitive compact Banach algebra in order that its representation given in § 2 is an isometry.

In § 4 we study a compact B* algebra and obtain a full representation theory for it. A primitive compact B* algebra is shown to be isometrically isomorphic to the algebra of all compact operators on some Hilbert space, and a compact B* algebra is shown to be the

$B(\infty)$ sum of a family of such algebras.

In § 5 we consider the operator ${}_aT_c$ on a compact Banach algebra A and obtain some conditions for ${}_aT_c$ to be a compact operator.

2. Some General Theorems.

DEFINITION 2.1. A Banach algebra A is said to be compact if, for each element a in A, the operator ${}_aT_a$ on A is compact.

We observe that a (left, right) completely continuous Banach algebra A is compact since ${}_aT_a = {}_aT\,T_a$. By Theorem 1.3.3 the algebra of compact operators on a Banach space X is compact, but this algebra is not (left, right) completely continuous unless X has finite dimension. Thus the class of compact algebras is considerably larger than the class of completely continuous algebras.

We now give some elementary lemmas.

LEMMA 2.2. Let A be a compact Banach algebra. Then

(i) if B is a closed subalgebra of A, B is a compact Banach algebra,

(ii) if I is a closed two-sided ideal of A, A/I is a compact Banach algebra.

Proof. The proof of (i) is trivial. To prove (ii) let $[a]$ be the coset $a + I$ in A/I. Then $\|[a]\| = \inf \|a'\|$, $a' \in [a]$. If $\{[b_n]\}$ is a sequence in A/I with $\|[b_n]\| \leqslant 1$, we may suppose that $\|b_n\| \leqslant 2$. Hence there exists a subsequence $\{b_{n_k}\}$ such that $\{ab_{n_k}a\}$ converges, and so, since the map $A \longrightarrow A/I$ is continuous, the sequence $\{[a][b_{n_k}][a]\}$ converges. Hence A/I is a compact Banach algebra.

LEMMA 2.3. Let A be a Banach algebra for which the operator ${}_aT_a$ is compact for all elements a of a dense subset of A. Then A is a compact Banach algebra.

Proof. Let S be this dense subset. For an element a in A there exists a sequence $\{a_n\}$ in S converging to a. For b in A we have

$$\|{}_aT_ab - {}_{a_n}T_{a_n}b\| = \|aba - a_nba_n\| = \|(a - a_n)ba + a_nb(a - a_n)\|$$
$$\leq \|a - a_n\|(\|a\| + \|a_n\|)\,\|b\|,$$

and hence ${}_{a_n}T_{a_n} \longrightarrow {}_aT_a$ as $n \to \infty$. Thus ${}_aT_a$ is compact and so A is a compact Banach algebra.

COROLLARY 2.4. A Banach algebra with a dense socle is a compact Banach algebra.

Proof. This follows from Lemma 2.3 and Theorem 3.3.3.

If $\{A_\lambda ; \lambda \in \Lambda\}$ is a family of Banach algebras, then $(\sum A_\lambda)_0$ denotes the subset of the full direct sum $\sum A_\lambda$ consisting of all elements f in $\sum A_\lambda$ such that, for arbitary $\varepsilon > 0$, the set $\{\lambda ; \|f(\lambda)\| \geq \varepsilon\}$ is finite. Then $(\sum A_\lambda)_0$ is a Banach algebra with the norm $\|f\| = \sup \|f(\lambda)\|$, $\lambda \in \Lambda$, and is called the $B(\infty)$ sum of the algebras A_λ $(\lambda \in \Lambda)$, — see (8), p. 411. It is easily seen that the set $S = \{f;\ f \in \sum A_\lambda,\ f(\lambda) = 0$ except on a finite subset of $\Lambda\}$ is a dense ideal of $(\sum A_\lambda)_0$.

LEMMA 2.5. If $A = (\sum A_\lambda)_0$ and each A_λ is a compact Banach algebra, then A is a compact Banach algebra.

Proof. Let S be the dense subset defined above. Then it is easily verified that ${}_aT_a$ is compact for each element a in S, and thus A is a compact Banach algebra by Lemma 2.3.

From Lemma 3.2.2 we have the following result.

LEMMA 2.6. Let a be an element of a compact Banach algebra A. Then 0 is the only possible accumulation point of Sp a, and if $\lambda \in \mathrm{Sp}\ a \setminus \{0\}$ there exists a non-zero element b in A such that b commutes with a and $ab = \lambda b$.

As a corollary of Lemma 2.6 we have the following result which will prove useful in the next two sections.

LEMMA 2.7. Let $a \longrightarrow \hat{a}$ be an isomorphism of a compact Banach algebra A into an arbitrary Banach algebra B. Then $\mathrm{Sp}\ a \setminus \{0\} = \mathrm{Sp}\ \hat{a} \setminus \{0\}$.

Proof. Clearly $\mathrm{Sp}\ a \setminus \{0\} \supseteq \mathrm{Sp}\ \hat{a} \setminus \{0\}$, and the opposite inclusion follows from the previous lemma.

From Theorem 4.3.2. we obtain the following representation theorem. If X and Y are Banach spaces in normed duality, denote by $K(X,Y,(\ , \ ))$ the subalgebra of $B(X,Y,(\ , \ ))$ consisting of all compact operators T on X for which T* is a compact operator on Y. This is a closed subalgebra of $B(X,Y,(\ , \ ))$ and is indeed a compact Banach algebra. For if $T \in K(X,Y,(\ , \ ))$ and $\{S_n\}$ is a sequence in $K(X,Y,(\ , \ ))$ such that $\|S_n\|' \leqslant 1$, there exists by Theorem 1.3.3 a subsequence $\{S_{n_k}\}$ and operators S in B(X), S' in B(Y), such that $\|TS_{n_k}T - S\| \longrightarrow 0$ and $\|T^*S_{n_k}^*T^* - S'\| \longrightarrow 0$. By the argument used in Theorem 4.2.3 S lies in $B(X,Y,(\ , \ ))$ and $S' = S^*$. Then $\|TS_{n_k}T - S\|' \longrightarrow 0$, and hence $K(X,Y,(\ , \ ))$ is a compact Banach algebra.

THEOREM 2.8. Let A be a primitive compact Banach algebra. Then there exist Banach spaces X and Y in normed duality and a norm-decreasing isomorphism $a \longrightarrow S_a$ of A into $K(X,Y,(\ , \ ))$ such that the socle of A is mapped onto $F(X,Y,(\ , \ ))$.

We now show that, in the primitive case, the existence of a non-zero central element distinguishes completely continuous Banach algebras in the class of compact Banach algebras.

THEOREM 2.9. Let A be a primitive compact Banach algebra. Then A contains a non-zero central element if and only if A is completely continuous.

Proof. A primitive completely continuous Banach algebra is homeomorphic to a full finite-dimensional matrix algebra, and thus contains a non-zero central element given by the identity matrix.

Conversely, suppose a is a non-zero central element in a primitive compact Banach algebra A. Then $\nu(a) > 0$ since otherwise a would lie in the radical of A. Then $\nu(a^2) > 0$, and ${}_{a^2}T$ is a compact operator on A. Then A is completely continuous by Theorem 2.3.3.

We now study the structure space of a compact Banach algebra, i.e. the space of primitive ideals with the hull-kernel topology.

THEOREM 2.10. The structure space of a compact Banach algebra A is discrete.

Proof. Since the structure space and the property of being compact do not alter on passage modulo the radical, we may assume that A is semi-simple.

If P is a primitive ideal in A, we first construct a non-zero idempotent which does not lie in P. Now A/P is a primitive compact Banach algebra and so contains a non-zero idempotent, which we write as $[a]$ for some element a in A, where $[a]$ is the coset a + P in A/P. Since $[a]$ is an idempotent, $1 \in Sp_{A/P}[a]$, and hence $1 \in Sp\ a$. Now ${}_aT|A(a)$ is a Riesz operator and hence $1 \in Sp\ {}_aT|A(a)$ — see Chapter III,

§2. Let $A(a) = N \oplus R$ be the corresponding decomposition of $A(a)$, where $N = \{b \in A(a);\ (a - 1)^{\nu} b = 0\}$ and $R = (a - 1)^{\nu} A(a)$ for some positive integer ν. Let e be the corresponding idempotent such that ${}_eT | A(a)$ is the projection onto N. If $b \in A(a)$, then $[b] \in \mathbb{C}[a]$ since $[a]$ is an idempotent, and if $b \in R$ then $[b] = 0$. Thus we have $\{[b];\ b \in N\} = \mathbb{C}[a] \neq (0)$, and thus $[e] \neq 0$ since $eb = b$ for $b \in N$. Hence $e \notin P$.

Now eAe is finite-dimensional and is semi-simple by Lemma 3.2.8. Thus by a theorem of Wedderburn eAe is the direct sum of a finite family of full finite-dimensional matrix algebras $\{M_i\}$, the primitive ideals of eAe being of the form $\underset{i \neq j}{\oplus} M_i$. Thus eAe has a discrete structure space.

Since $e \notin P$, $P \not\supseteq eAe$ and so $P \cap eAe$ is a primitive ideal in eAe by Lemma 3.2.8. Again by this lemma, since eAe has a discrete structure space, there exists b in eAe such that $b \notin P \cap eAe$ but $b \in P' \cap eAe$ for every primitive ideal P' of A such that $P' \not\supseteq eAe$ and $P' \neq P$. Thus $b \notin P$ but $b \in P'$ for all primitive ideals P' of A except P. Thus $\{P\}$ is an open set in the structure space of A and, since P is an arbitrary primitive ideal of A, it follows that the structure space of A is discrete.

COROLLARY 2.11. A completely continuous Banach algebra has a discrete structure space.

COROLLARY 2.12. A Banach algebra with a dense socle has a discrete structure space.

Proof. This follows from Corollary 2.4 and Theorem 2.10.

COROLLARY 2.13. (0) is the only primitive ideal of a primitive compact Banach algebra.

We now give two results which give lower and upper bounds to the size of a semi-simple compact Banach algebra.

Let **A** be a semi-simple compact Banach algebra with primitive ideals P_λ $(\lambda \in \Lambda)$, and let $A_\lambda = \mathbf{A}/P_\lambda$. Let a_λ be the canonical image of a in $\mathbf{A}_\lambda$, and let $\|a_\lambda\|$ be the usual infimum norm, $\|a_\lambda\| = \inf \|a'\|$, $a' \in a + P_\lambda$. Each A_λ is a primitive compact Banach algebra, and thus there exists a norm-decreasing isomorphism of A_λ into $K(X_\lambda, Y_\lambda, (\ ,\ ))$ for some pair of Banach spaces X_λ and Y_λ in normed duality, where the image of $\mathbf{A}_\lambda$ contains $F(X_\lambda, Y_\lambda, (\ ,\ ))$. Let $\hat{a}(\lambda)$ be the image of a_λ under this isomorphism. Then the fuction $\hat{a}$ lies in the normed full direct sum $\sum K(X_\lambda, Y_\lambda, (\ ,\ ))$, since $\|\hat{a}(\lambda)\| \leq \|a_\lambda\| \leq \|a\|$. Thus $\|\hat{a}\| = \sup \|\hat{a}(\lambda)\| \leq \|a\|$. Thus we have a norm-decreasing isomorphism of **A** into $\sum K(X_\lambda, Y_\lambda, (\ ,\ ))$. Denote by F the socle of $\sum K(X_\lambda, Y_\lambda, (\ ,\ ))$. It is easily seen that F consists of all elements f in $\sum K(X_\lambda, Y_\lambda, (\ ,\ ))$ such that $f(\lambda)$ has finite rank for each λ in Λ and $f(\lambda) = 0$ for all but a finite subfamily of Λ. If F_λ is the set of elements f in $\sum K(X_\lambda, Y_\lambda, (\ ,\ ))$ such that $f(\lambda) \in F(X_\lambda, Y_\lambda, (\ ,\ ))$, $f(\mu) = 0$ $(\mu \neq \lambda)$, then $F = \sum F_\lambda$.

THEOREM 2.14. Under the isomorphism $a \longrightarrow \hat{a}$ the socle of **A** is mapped onto F.

Proof. By Corollary 3.2.10 the socle of **A** exists. Take some fixed μ in Λ. Then $\bigcap_{\lambda \neq \mu} P_\lambda \neq (0)$ since the structure space of **A** is discrete. There exists a two-sided ideal I_μ of $\mathbf{A}_\mu$ such that $a \in \bigcap_{\lambda \neq \mu} P_\lambda$ if and only if $a_\lambda = 0$ for $\lambda \neq \mu$ and $a_\mu \in I_\mu$. Then $I_\mu \neq (0)$ and hence the image of I_μ in $K(X_\mu, Y_\mu, (\ ,\ ))$ must contain $F(X_\mu, Y_\mu, (\ ,\ ))$, since the image of $\mathbf{A}_\mu$ contains $F(X_\mu, Y_\mu, (\ ,\ ))$. Thus F lies in the image of **A** under the isomorphism $a \longrightarrow \hat{a}$. If $\hat{a}$ is a minimal idempotent in $\sum K(X_\lambda, Y_\lambda, (\ ,\ ))$, then a is a minimal idempotent in **A**. Thus, if $\hat{a}$

lies in F, we may write $a = \sum_{i=1}^{n} b_i e_i$ for some elements b_i and e_i $(1 \leq i \leq n)$ in A, where $\hat{b}_i \in F$ and e_i is a minimal idempotent of A. Thus a lies in the socle of A. Conversely, if a is a minimal idempotent of A, then $\hat{a}\hat{b}\hat{a} \in \mathbb{C}\hat{a}$ for all b in A, and by taking $\hat{b}$ in F it is clear that $\hat{a}$ is a minimal idempotent of $\sum K(X_\lambda, Y_\lambda, (\ ,\ ))$. Thus the image in $\sum K(X_\lambda, Y_\lambda, (\ ,\ ))$ of the socle of A is F.

Theorem 2.14 gives us a lower bound to the size of A. The next theorem gives us a very rough upper bound.

THEOREM 2.15. Given an element a in A and $\varepsilon > 0$, then

$$\|a_\lambda b_\lambda a_\lambda\| \leq \varepsilon \quad \text{for b in A such that } \|b\| \leq 1,\ \hat{b} \in F_\lambda,$$

except on a finite subset of Λ. In particular, $a_\lambda = 0$ except on a countable subset of Λ.

Proof. Suppose the result is not true. Suppose for some $\varepsilon > 0$ and some infinite subset Λ' of Λ that for each μ in Λ' there exists a point b^μ in A such that $\widehat{b^\mu} \in F_\mu$, $\|b^\mu\| \leq 1$, and $\|a_\mu b^\mu_\mu a_\mu\| \geq \varepsilon$. Then if $\mu \neq \mu'$

$$\begin{aligned}\|a(b^\mu - b^{\mu'})a\| &\geq \sup \|a_\lambda(b^\mu_\lambda - b^{\mu'}_\lambda)a_\lambda\| \\ &= \max\{\|a_\mu b^\mu_\mu a_\mu\|,\ \|a_{\mu'} b^{\mu'}_{\mu'} a_{\mu'}\|\} \\ &\geq \varepsilon.\end{aligned}$$

Thus there exists no subsequence $\{b^{\mu_k}\}$ such that $\{ab^{\mu_k}a\}$ converges. Since A is a compact Banach algebra we have a contradiction.

It follows that $a_\lambda b_\lambda a_\lambda = 0$ for all b in A with $\hat{b} \in F_\lambda$ except on a countable subset of Λ. Hence $a_\lambda = 0$ except on this countable subset.

3. The Approximate $B^{\#}$ Condition.

We now place additional conditions on a primitive compact Banach algebra in order that the representation given in Theorem 2.8 is an isometry.

DEFINITION 3.1. A Banach algebra A is said to satisfy the approximate $B^{\#}$ condition if, given a in A and $\varepsilon > 0$, there exists an element $a^{\#}$ in A (not necessarily unique) such that $\|a^{\#}\| = 1$ and $\nu(a^{\#}a) \geq \|a\| (1 - \varepsilon)$.

This definition is due to Smiley (14) although he does not give it a name, and is a slight generalisation of the definition of a $B^{\#}$ algebra given by Bonsall (2), in which he requires that $\nu(a^{\#}a) = \|a\|$.

We need the following piece of notation.

If the Banach spaces X and Y are in normed duality we write

$$\|y\|_0 = \sup_{\|x\| \leq 1} |(x,y)|, \quad \|x\|_0 = \sup_{\|y\| \leq 1} |(x,y)| .$$

We remind the reader that the norm on $K(X,Y,(\ , \ ))$ is given by $\|T\|' = \max \{ \|T\|, \|T^*\| \}$.

THEOREM 3.2. Let A be a primitive compact Banach algebra satisfying the approximate $B^{\#}$ condition, and let $a \longrightarrow S_a$ $(a \in A)$ be the isomorphism of A into $K(X,Y,(\ , \ ))$ given by Theorem 2.8. Then for all $a \in A$, $x \in X$, $y \in Y$, we have $\|a\| = \|S_a\| = \|S_a{}^*\|$ and $\|x\| = \|e\| \, \|x\|_0$, $\|y\| = \|e\| \, \|y\|_0$.

Proof. We remind the reader that e is a minimal idempotent of A, $X = Ae$, $Y = eA$, $S_a x = ax$ $(x \in X)$, and $S_a{}^* y = ya$ $(y \in Y)$. By Lemma 2.7 $\nu(a) = \nu_{B(X)}(S_a)$ for all a in A. Given a in A and $\varepsilon > 0$, there exists an element $a^{\#}$ in A such that $\|a^{\#}\| = 1$ and $\nu(a^{\#}a) \geq \|a\| (1 - \varepsilon)$. Since $\|S_{a^{\#}}\| \leq \|a^{\#}\| = 1$, we therefore have

$\|a\|(1-\varepsilon) \leqslant V(a^{\#}a) = V_{B(X)}(S_{a^{\#}}S_a) \leqslant \|S_{a^{\#}}S_a\| \leqslant \|S_a\| \leq \|a\|.$

As ε is arbitrary this gives $\|a\| = \|S_a\|$, and similarly $\|a\| = \|S_a*\|$.

Taking a to be of the form xy $(x \in X,\ y \in Y)$, we have

$\|x\|\,\|y\|_0 = \|x \boxtimes y\| = \|S_{xy}\| = \|xy\| = \|S_{xy}^*\| = \|y \boxtimes x\| = \|x\|_0\,\|y\|.$

Hence there exists a constant $k \neq 0$ such that $\|x\| = k\,\|x\|_0$ and $\|y\| = k\,\|y\|_0$ for all x in X, y in Y. Now $e \in Y$ and $\|e\|_0 = 1$ since, for x in X,

$$\|x\|\,\|e\|_0 = \|S_{xe}\| = \|xe\| = \|x\|$$

as $xe = x$. Thus $k = \|e\|$ and the proof is complete.

COROLLARY 3.3. The isomorphism $a \longrightarrow S_a$ is an isometry of A into K(X,Y,(,)).

The fact that there exists a positive scalar k such that $\|x\| = k\,\|x\|_0$ and $\|y\| = k\,\|y\|_0$ should come as no suprise, since, if we impose on A a condition that ensures that the mapping $a \longrightarrow S_a$ of A into K(X,Y,(,)) is an isometry, then we may reasonably expect that $\|S_a\| = \|S_a*\| = \|a\|$ since $\|S_a\|' = \max\{\|S_a\|, \|S_a*\|\}$. In particular $\|x \boxtimes y\| = \|(x \boxtimes y)^*\| = \|y \boxtimes x\|$ for all x in X, y in Y, and hence such a scalar k exists. If such a k exists we have the converse that $\|T\| = \|T^*\|$ for T in B(X,Y,(,)), since

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\| \leq 1}\left(\sup_{\|y\| \leq 1} k\,|(Tx,y)|\right) = \sup_{\|y\| \leq 1}\left(\sup_{\|x\| \leq 1} k\,|(x,T^*y)|\right)$$
$$= \sup_{\|y\| \leq 1} \|T^*y\| = \|T^*\|.$$

The following theorem shows that the approximate $B^{\#}$ condition is a most natural anl economical condition to choose in order to ensure that the isomorphism of A into K(X,Y,(,)) is an isometry.

THEOREM 3.4. Let X and Y be Banach spaces in normed duality such that $\|x\| = k\|x\|_0$ and $\|y\| = k\|y\|_0$ $(x \in X,\ y \in Y)$ for some positive scalar k. Then any closed subalgebra of $B(X,Y,(\ ,\ ))$ containing $F(X,Y,(\ ,\ ))$ satisfies the approximate $B^{\#}$ condition.

Proof. As we remarked before $\|T\| = \|T\|' = \|T^*\|$ for T in $B(X,Y,(\ ,\ ))$. Let A be a closed subalgebra of $B(X,Y,(\ ,\ ))$. Given T in A and $\varepsilon > 0$ choose x in X such that $\|x\| = 1$ and $\|Tx\| \geq \|T\|(1-\varepsilon)^{1/2}$, and choose y in Y such that $\|y\| = k$ and $|(Tx,y)| \geq k\|Tx\|_0(1-\varepsilon)^{1/2}$. Hence $\|y\|_0 = 1$ and $|(Tx,y)| \geq \|Tx\|(1-\varepsilon)^{1/2}$. Now $x \otimes y \in A$, $\|x \otimes y\| = \|x\|\,\|y\|_0 = 1$, and

$$\nu\big((x \otimes y)T\big) = |(Tx,y)| \geq \|Tx\|(1-\varepsilon)^{1/2} \geq \|T\|(1-\varepsilon).$$

Thus, taking $T^{\#}$ to be $x \otimes y$, we have the required result.

NOTE. Examples of such spaces abound, the most obvious examples being when either $Y = X^*$ or $X = Y^*$.

One of the outstanding conjectures still to be resolved in the theory of compact operators is the following;

I) On a Banach space X is each compact operator the limit in the operator norm of a sequence of finite rank operators on X?

We now pose a second conjecture which contains the above conjecture but does not appear to be equivalent to it.

II) If X and Y are a pair of Banach spaces in normed duality such that, for some positive k, $\|x\| = k\|x\|_0$ and $\|y\| = k\|y\|_0$ $(x \in X,\ y \in Y)$, is $K(X,Y,(\ ,\ ))$ the closure of $F(X,Y,(\ ,\ ))$ in $B(X,Y,(\ ,\ ))$?

The conditions in II) between the spaces X and Y are satisfied if $Y = X^*$ and in this case II) reduces to I).

If conjecture II were true we would have the following result.

If A is a primitive compact Banach algebra satisfying the approximate $B^{\#}$ condition, then there exist a pair of Banach spaces X and Y in normed duality such that A is isometrically isomorphic to K(X,Y,(,)).

This follows from Theorem 3.2 and the fact that the image of A contains F(X,Y,(,)).

Partial results can be obtained in the case of a compact Banach algebra which satisfies the approximate $B^{\#}$ condition but is not necessarily primitive. They are not as complete as the results obtained by Olubummo (11) for a left completely continuous $B^{\#}$ algebra, and we do not include them. Olubummo proves that a Banach algebra A is a left completely continuous $B^{\#}$ algebra if and only if A is the $B(\infty)$ sum of simple finite-dimensional $B^{\#}$ algebras. A simple finite-dimensional $B^{\#}$ algebra is isometrically isomorphic to a finite-dimensional matrix algebra with the operator norm.

4. B* algebras.

DEFINITION 4.1. A Banach *-algebra is defined to be a Banach algebra A with an involution $a \longrightarrow a^*$ satisfying

(i) $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$

(ii) $(ab)^* = b^* a^*$

(iii) $a^{**} = a$

for all a and b in A and scalars α and β. If in addition

(iv) $\|aa^*\| = \|a\|^2$

for all a in A, then A is a B* algebra.

We need the following result from (12), Lemma 4.10.1.

LEMMA 4.2. Let A be an arbitrary *-algebra in which $x^*x = 0$ implies $x = 0$. Then every minimal left ideal L of A is of the form $L = Ae$ where e is a unique hermitian idempotent.

Now let A be a primitive *-algebra with minimal ideals such that, if $x \in A$ and $x^*x = 0$, then $x = 0$. If L is a minimal left ideal of A there exists a hermitian idempotent e such that $L = Ae$. As L is minimal $eAe = \mathbb{C}e$ by (12), Corollary 2.1.6. If x and y lie in L then $y^*x \in eAe$, and thus we can define a scalar (x,y) by

$$y^*x = (x,y)e \qquad (x \in L,\ y \in L).$$

Now (x,y) is clearly linear in x for fixed y, and also

$$(y,x)e = x^*y = (y^*x)^* = ((x,y)e)^* = \overline{(x,y)}e.$$

Hence $(y,x) = \overline{(x,y)}$. Also, since $(x,x)e = x^*x$, $(x,x) = 0$ implies that $x = 0$. Hence by a standard argument (x,x) is real and has the same sign for all x in X. Since e lies in L and $e^*e = e$, it follows that $(e,e) = 1$, and so (x,x) is positive for all x in L. Thus (x,y) is an inner product on L.

We define a second norm on L by

$$|x| = (x,x)^{1/2} \qquad (x \in L).$$

In (12), Theorem 4.10.6, it is shown that L is complete in the inner product norm if and only if there exists a constant k such that $\|x\|^2 \leq k\,\|x^*x\|$ for every x in L. The proof depends upon the continuity of the involution and upon the closed graph theorem, and shows that the inner product norm is equivalent to the norm on L induced by A. In particular, if there exists a constant k such that $\|a\|^2 \leq k\,\|a^*a\|$ for all a in A, then L is complete relative to the norm $|x|$, which is equivalent to the norm $\|x\|$, and thus L is a Hilbert space. Following convention we shall now denote by H the Banach space L regarded as a Hilbert space. Denote by $|\cdot|$ the operator norm on B(H).

We are now in a position to prove the main result of this section.

THEOREM 4.3. Let A be a Banach *-algebra such that $\|a\|^2 \leq k\|a^*a\|$ $(a \in A)$ for some constant k. If A is primitive and compact there exists a *-homeomorphism of A onto the algebra of compact operators on some Hilbert space. If A is a B* algebra the homeomorphism is an isometry.

Proof. We can construct a Hilbert space H as on the previous page since A, being compact, has minimal ideals. We take the representation of A on H given by $a \longrightarrow S_a$ $(a \in A)$, where $S_a x = ax$ $(x \in H)$. The representation is faithful since A is primitive, and is clearly continuous. Since

$$(S_a x, y)e = y^*ax = (a^*y)^*x = (x, S_{a^*}y)e \qquad (x \in H,\ y \in H,\ a \in A),$$

it is a *-representation. If x, y, and z lie in H then

$$S_{yz^*}(x) = yz^*x = (x,z)y = y \otimes z\ (x),$$

and so the image of A in B(H) contains all finite rank operators on H.

If h is a hermitian element of A (i.e. $h^* = h$), the condition on the norm of A ensures that $\|h\| \leq k\,\nu(h)$. By Lemma 2.7 we have $\nu(a) = \nu_{B(H)}(S_a)$ for a in A. Hence for a in A

$$\|a\|^2 \leq k\,\|a^*a\| \leq k^2\nu(a^*a) = k^2\nu(S_a^* S_a) = k^2\,|S_a|^2.$$

Thus the mapping $a \longrightarrow S_a$ $(a \in A)$ is a *-homeomorphism of A into B(H).

As in Theorem 4.3.2 the operator ${}_aT_e$ is a compact operator on A, and hence S_a is compact for each a in A. Since the operators of finite rank are dense in the algebra of all compact operators on a Hilbert space, it follows that the mapping $a \longrightarrow S_a$ $(a \in A)$ maps A onto the algebra of all compact operators on H.

If A is a B* algebra, the mapping is an isometry, since a B* algebra has a unique norm with the B* property.

THEOREM 4.4. A compact B* algebra A is a $B(\infty)$ sum of algebras A_λ, $A = (\sum A_\lambda)_0$, where each A_λ is the algebra of all compact operators on a Hilbert space H_λ. Conversely, each such algebra A is compact.

Proof. Let $\{P_\lambda ; \lambda \in \Lambda\}$ be the family of primitive ideals of A. If $A_\lambda = A/P_\lambda$ $(\lambda \in \Lambda)$, then A_λ is a primitive compact B* algebra, and by Theorem 2.10 the structure space of A is discrete. By Theorem 4.9.24 in (12) A has a discrete structure space if and only if $A = (\sum A_\lambda)_0$. Hence by the previous theorem the required result follows.

The converse is immediate from Lemma 2.5.

COROLLARY 4.5. A B* algebra is compact if and only if it has a dense socle.

Proof. The socle of the algebra of compact operators on a Hilbert space is dense, being the ideal of finite rank operators. Thus, if A is a compact B* algebra, it is clear from the previous theorem that A has a dense socle. Conversely, by Corollary 2.4, a B* algebra with a dense socle is compact.

For the next corollary we need the following notation and definition due to Kaplansky (7).

If E is an arbitrary subset of an algebra A, let

$$A_l(E) = \{x;\ x \in A,\ xE = (0)\}$$
$$A_r(E) = \{x;\ x \in A,\ Ex = (0)\}.$$

An algebra A is called a dual algebra if, for arbitrary closed left ideal L and closed right ideal R in A,

$$A_l A_r(L) = L, \quad A_r A_l(R) = R.$$

In the case of a B* algebra the concept of a dual algebra coincides with the weaker concept of an annihilator algebra introduced by Bonsall and Goldie in (5). On gathering together the results of Theorem 4.10.14, Corollary 4.10.20, and Theorem 4.10.25 in (12) we have;

A B* algebra A is dual if and only if it is the $B(\infty)$ sum of its family of minimal-closed two-sided ideals $\{A_\lambda\}$ each of which is isometric to the algebra of all compact operators on some Hilbert space H_λ.

Thus we have the following corollary.

COROLLARY 4.6. A B* algebra is compact if and only if it is dual.

As a second corollary we obtain a result due to Kaplansky (10), Theorem 2.1.

COROLLARY 4.7. A B* algebra is dual if and only if it has a dense socle.

5. The Operator ${}_aT_c$.

In a semi-simple Banach algebra we have proved that, if ${}_aT_a$ and ${}_cT_c$ are Riesz operators, then so is ${}_aT_c$. We now examine this problem when ${}_aT_a$ is compact for every a in A. We can not prove that ${}_aT_c$ is compact, but we can prove that ${}_aT_c^2$ is compact. We need the following piece of notation.

Let a and b be elements of an associative ring A. We denote by a.b the Jordan product $a.b = ab + ba$, and denote by A.A the square of A under the Jordan product.

THEOREM 5.1. Let A be a compact Banach algebra. Then the operators ${}_aT_c$ and ${}_cT_a$ are compact for all a in A and all c in $\overline{A.A}$.

Proof. For a and c in A the operators ${}_{a+c}T_{a+c}$ and ${}_{a-c}T_{a-c}$ are compact.

Hence by subtraction $({}_aT_c + {}_cT_a)$ is a compact operator. Since

$$ {}_aT_{c^2} = T_c({}_aT_c + {}_cT_a) - {}_cT_c\,T_a , $$

it follows that ${}_aT_{c^2}$ is compact for all elements a and c in **A**. For b, d in **A**, $(b + d)^2 - (b - d)^2 = 2(b.d)$, and hence ${}_aT_c$ is compact if c lies in **A.A**. Then ${}_aT_c$ is compact for all a in **A** and all c in $\overline{\mathbf{A.A}}$.

Similarily, ${}_cT_a$ is compact for all a in **A** and all c in $\overline{\mathbf{A.A}}$.

COROLLARY 5.2. For a compact Banach algebra **A** the operator ${}_aT_c^2$ is compact for all a and c in **A**.

We now examine the set **A.A**.

LEMMA 5.3. If **A** is an associative ring over a field of characteristic not equal to two, then $\mathbf{A.A} \supseteq A^3$.

Proof. Let $a_i \in A$ $(i = 1,2,3)$. Then we have

$$ (a_1a_2).a_3 = a_1a_2a_3 + a_3a_1a_2 $$

$$ (a_3a_1).a_2 = a_3a_1a_2 + a_2a_3a_1 $$

$$ (a_2a_3).a_1 = a_2a_3a_1 + a_1a_2a_3. $$

Hence $(a_1a_2).a_3 - (a_3a_1).a_2 + (a_2a_3).a_1 = 2(a_1a_2a_3)$, and so we have $A^3 \subseteq \mathbf{A.A}$.

COROLLARY 5.4. **A.A** is a two-sided ideal of **A**.

COROLLARY 5.5. If **A** is a Banach algebra with a socle S, then $S \subseteq \mathbf{A.A}$.

Proof. Any idempotent of **A** will lie in **A.A**, and thus the result follows from the previous corollary.

If $\overline{A^3} = A$, then $\overline{\mathbf{A.A}} = \mathbf{A}$, and so ${}_aT_c$ is compact for all a and c in **A**. In particular, this occurs if **A** has an approximate identity, i.e. there exists a family $\{e_\lambda ; \lambda \in \Lambda\}$ of elements of A, where Λ is a directed set, such that $\|e_\lambda\| = 1$ $(\lambda \in \Lambda)$ and

$$\lim e_\lambda a = \lim a e_\lambda = a$$

for each a in **A**. By Theorem 4.8.14 (12) a B* algebra has an approximate identity, and thus for a B* algebra $\overrightarrow{\mathbf{A}.\mathbf{A}} = \mathrm{A}$.

We do not have an example of a compact Banach algebra for which ${}_aT_c$ is not a compact operator for some a and c in **A**. For the algebra **A** consisting of the compact operators on a Banach space the operator ${}_aT_c$ is, of course, compact for all a and c in **A**.

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